Between Nth- and $(N+1)$th-Degree Stochastic Dominance

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Under an expected utility framework, we develop a continuum of stochastic dominance rules for individuals whose Nth-degree absolute risk aversion has a common negative lower bound. We justify the preference constraint by extending the concept of uniform risk aversion proposed by Aumann and Serrano (2008) to higher orders. The new rules encompass the standard Nth-degree stochastic dominance, and enlarge the class of comparable risk changes that are subject to the notion of risk apportionment pioneered by Eeckhoudt and Schlesinger (2006). We generalize the existing results regarding stochastic dominance, and illustrate the applications of our rules in various important decision problems.

Key words: stochastic dominance, risk aversion, higher-order risk preferences, precautionary effects, risk-taking, social welfare.

1. Introduction

Stochastic dominance rules have served as a cornerstone in the literature on risk theory (Bawa 1982, Levy 1992). They are the consensus rules to rank distributions for all individuals in specific sets. For example, first-degree stochastic dominance (FSD) corresponds to all non-satiable decision makers. If individuals are not only non-satiable but also risk-averse, then second-degree stochastic dominance (SSD) can be applied to rank distributions for all of these individuals. For non-satiable, risk-averse and prudent individuals, third-degree stochastic dominance (TSD) is the corresponding distribution ranking criterion. In general, the ranking criterion for individuals exhibiting mixed-risk aversion up to degree $N \in \mathbb{N}$ is termed Nth-degree stochastic dominance (NSD).

Although integer-degree stochastic dominance has been widely studied and applied, the jumps on the preference assumptions from FSD to SSD, SSD to TSD and so on are substantial. As is well-known in the classical literature, the risk preference of a non-satiable individual may not always exhibit risk aversion within the range of the wealth domain. For example, Friedman and Savage (1948) proposed a utility function that was concave for small and large wealth and convex for wealth in between.[1]

[1] He explained that individuals with this type of utility would simultaneously purchase insurance and lottery tickets. Jullien and Salanie (2000) further found that the Friedman-Savage type of utility could better explain the behavior of bettors in British horse races.

The recent experimental findings also suggest that dissatisfaction exists with the integer-degree stochastic dominance rules and sustain the needs of the continua among them. The trends in the experimental literature to estimate individuals' higher-order preferences were blew up by a seminal paper by Eeckhoudt and Schlesinger (2006) which characterizes prudence, temperance and higher-order preferences over simple lotteries. Based on the experiments, the higher-order preferences can be observed and thus one can properly apply the corresponding stochastic dominance rule for a set of individuals revealing the same types of preferences. However, the experimental evidence suggests that only a minority of the subjects always make prudent or temperate choices among all the tasks. For example, Deck and Schlesinger (2010) found that 84% (70%) of the subjects made prudent and imprudent (temperate and intemperate) choices at the same time. Only 14% (6%) were prudent (temperate) in all experimental tasks. Ebert and Wiesen (2011) also obtained similar results. Only 8% of the subjects acted prudently among all of the lottery choices in their experiments. 92% of the subjects chose at least one of the imprudent choices.[2]

In this paper, we search for the consensus rules to rank distributions for new sets of individuals which interpolate the integer-degree stochastic dominance. The idea of interpolating integer-degree stochastic dominance rules has been developed in two distinct ways in the literature. Fishburn (1976) employs fractional calculus to derive the continua of stochastic dominance rules, and Müller et al. (2016) develop a stochastic dominance rule between FSD and SSD through adding constraints to the ratio of marginal utilities. Our paper fulfills the same mission with an alternative approach. As we will explore in more detail later, the relative strength of our approach lies in its coincidence with the uniform comparative statics on risk attitudes, its compatibility with the notion of the risk apportionment of all orders, as well as its amenability to comparative analysis. More importantly, under the proposed continuum, many existing definitions and results can be seen as special cases.

While the integer-degree stochastic dominance rules only consider the sign of the derivatives of the utility functions, we interpolate these rules by adding the information on the intensity of such risk attitudes. Specifically, we assume that, in the context of expected utility, the utility function u exhibits mixed-risk aversion up to degree N (Caballé and Pomansky 1996), and the associated $(N+1)$th-degree index of absolute risk aversion is no less than $(1-1/c)$, where $c \in (0,1)$.[3] Although we still use the term "risk aversion" in the $(N+1)$th-degree index of risk aversion, it is allowed to be negative. For example, if $N=1$, then the preferences of the individuals exhibit $u' > 0$ and

[2] The literature also finds that risk-averse individuals could be either prudent or imprudent. For example, Deck and Schlesinger (2014) found that the correlation between the percentage of choices of risk aversion and that of prudence was insignificant. Noussair et al. (2014) found a significant positive correlation between risk aversion and prudence. However, in their laboratory sample, this correlation was not significant.

[3] In Section 5, we show how to extend our findings when the information on the lower bound of the degrees of relative risk aversion is available.

$-\frac{u''}{u'} \geq 1 - \frac{1}{c}$. Any individuals in this set are non-satiable. They could be risk-averse or risk-loving but they are not too risk-loving.[4] If $N = 2$, then the preferences of the individuals exhibit $u' > 0$, $u'' < 0$ and $-\frac{u'''}{u''} \geq 1 - \frac{1}{c}$. That is, the individuals are non-satiable and risk-averse. They could be either prudent or imprudent, but they cannot be too imprudent. We propose the distribution ranking criteria for such individuals, and refer to these rules as $(N + c)$th-degree stochastic dominance, or notationally (N+c)SD.

We justify the constraint on the $(N + 1)$th-degree index of absolute risk aversion by adopting the concept of uniform risk-aversion proposed by Aumann and Serrano (2008) in Section 2. Section 3 provides the integral conditions for (N+c)SD. It extends and generalizes the relationship between NSD and risk apportionment revealed by Eeckhoudt et al. (2009b) to (N+c)SD. Section 4 offers alternative characterizations for (1+c)SD based on the existing results for FSD and SSD. It investigates (1+c)SD in terms of probability transfer, the single-crossing property, and affine transformation, providing new insights into the choice behavior of individuals with a limited degree of risk lovingness. Section 5 discusses how our approach can be adopted to generalize convex stochastic dominance and how to interpolate stochastic dominance with relative risk aversion. Section 6 illustrates that (N+c)SD is amenable to the comparative statics exercises for a broad set of economic problems such as precautionary saving and self-protection. We further show that the concept of (N+c)SD can be applied to provide a continuum of expectation dependence relationships which were first established by Wright (1987) and extended to higher integer orders by Li (2011). The corresponding application to the risky investment problem is demonstrated. We also provide an application to social inequality. Proofs are relegated to the Appendix.

2. Comparison of Nth-Degree Risk Aversion

In this section, we provide a rationale as to why we employ the lower bound of the index of absolute risk aversion as a major tool to classify individuals. To build up the economic foundation of the rationale, we compare the intensity of higher-order preferences among individuals based on the choices of simple lottery pairs which are adopted to characterize the sign of the derivatives of the utility function in Eeckhoudt and Schlesinger (2006). Our approach extends and generalizes the concept of "uniform no less risk aversion" introduced by Aumann and Serrano (2008) to higher-order preferences. In addition, our analyses are not restricted to mixed risk-averse individuals. Risk-loving, imprudent, or intemperate individuals can be included in our analyses.

[4] The distribution ranking rules based on cumulative prospect theory (Kahneman and Tversky 1979) offer another approach in identifying risk changes for decision makers whose preferences could be both risk-averse and risk-loving. Levy and Wiener (1998) develop prospect stochastic dominance assuming S-shaped utility functions and nonlinear probability weighting functions. Baucells and Heukamp (2006) extend SSD conditions to the case where individuals could be risk-averse in some ranges and risk-loving in others. Our approach differs from the above as we only require the individuals' extreme of being risk-loving to be known in the whole domain. Furthermore, our approach accommodates all higher orders such as cases where the individuals are sometimes prudent and sometimes imprudent.

Let u denote a von Neumann-Morgenstern utility function for money defined over $\mathbb{R}$. It is continuously differentiable up to order $N \in \mathbb{N}$. To unify the notation, in the following, we use $u^{(N)}$ to denote the Nth-order derivative of u. In the literature, $u^{(1)} > 0$ captures non-satiability; $u^{(2)} < 0$ is the well-accepted condition for risk aversion (Pratt 1964); $u^{(3)} > 0$ is known as "prudence" as it captures an individual's motive for precautionary saving (Kimball 1990); and $u^{(4)} < 0$ is known as "temperance" in that it captures the preference for disaggregating two independent pure risks (Kimball 1993).

The "building blocks" for the analysis are defined by iteration. Let $k > 0$ denote a constant and $\tilde{x}$ denote a non-degenerate gamble. $\tilde{x}$ is allowed to have either a positive, zero or negative mean, and becomes a *pure risk* if $E[\tilde{x}] = 0$. Given two gambles, $[\tilde{x}_1; \tilde{x}_2]$ denotes the 50-50 lottery in which $\tilde{x}_1$ and $\tilde{x}_2$ are realized with equal probability. Define

$$A_1 = -k, \quad B_1 = 0; \quad A_2(\tilde{x}) = \tilde{x}, \quad B_2(\tilde{x}) = 0;$$
$$A_3(\tilde{x}) = [B_1; \tilde{x} + A_1], \quad B_3(\tilde{x}) = [A_1; \tilde{x} + B_1].$$

Here, A_1 represents a sure loss in wealth of size k, A_2 represents a risky gamble, and A_3 attaches the gamble to the loss state. B_1 and B_2 signify a normal riskless state, and B_3 attaches the gamble to the normal state. For $N \geq 4$, let $\lfloor y \rfloor$ denote the greatest integer not exceeding the real number y, and $\{\tilde{\varepsilon}_n\}_{n=1}^{\lfloor N/2 \rfloor - 1}$ be an indexed set of non-degenerate zero-mean pure risks that are not only mutually independent but also independent of $\tilde{x}$. Define

$$A_N\left(\tilde{x}, \tilde{\varepsilon}_1, \cdots, \tilde{\varepsilon}_{\lfloor N/2 \rfloor - 1}\right) = \left[B_{N-2}\left(\tilde{\varepsilon}_1, \cdots, \tilde{\varepsilon}_{\lfloor N/2 \rfloor - 1}\right); \tilde{x} + A_{N-2}\left(\tilde{\varepsilon}_1, \cdots, \tilde{\varepsilon}_{\lfloor N/2 \rfloor - 1}\right)\right],$$
$$B_N\left(\tilde{x}, \tilde{\varepsilon}_1, \cdots, \tilde{\varepsilon}_{\lfloor N/2 \rfloor - 1}\right) = \left[A_{N-2}\left(\tilde{\varepsilon}_1, \cdots, \tilde{\varepsilon}_{\lfloor N/2 \rfloor - 1}\right); \tilde{x} + B_{N-2}\left(\tilde{\varepsilon}_1, \cdots, \tilde{\varepsilon}_{\lfloor N/2 \rfloor - 1}\right)\right],$$

where A_N attaches the gamble $\tilde{x}$ to the "bad" state A_{N-2} and B_N attaches $\tilde{x}$ to the "good" state B_{N-2}. To simplify the notation, we abbreviate $A_N\left(\tilde{x}, \tilde{\varepsilon}_1, \cdots, \tilde{\varepsilon}_{\lfloor N/2 \rfloor - 1}\right)$ and $B_N\left(\tilde{x}, \tilde{\varepsilon}_1, \cdots, \tilde{\varepsilon}_{\lfloor N/2 \rfloor - 1}\right)$ as $A_N(\tilde{x})$ and $B_N(\tilde{x})$, respectively.

Based on the choices of the above lottery pairs, Eeckhoudt and Schlesinger (2006, p. 284) define risk apportionment of order N as follows:

Definition 1 (Eeckhoudt and Schlesinger (2006)). For $N \in \mathbb{N}$, preferences are said to satisfy **risk apportionment of order** N, if for any zero-mean gamble $\tilde{x}$, the individual always prefers $B_N(\tilde{x})$ to $A_N(\tilde{x})$ at all levels of initial wealth.

They further show that $(-1)^{N+1} u^{(N)} > 0$ is equivalent to risk apportionment of order N.

Extending their approach, we proceed one step further to explore how to compare the "intensity of risk apportionment" across individuals. To generalize the comparison to the situation where $(-1)^{N+1} u^{(N)}$ takes negative values somewhere, we allow the gamble $\tilde{x}$ to have arbitrary means. Motivated by Aumann and Serrano (2008), we define "uniformly no less intensity of risk apportionment of order N" as follows:

Definition 2. For $N \in \mathbb{N}$, we say that individual i exhibits **uniformly no less intensity of risk apportionment of order** N than individual j, written as $i \trianglerighteq_N j$, if for any gamble $\tilde{x}$, whenever i prefers $A_N(\tilde{x})$ to $B_N(\tilde{x})$ at some level of wealth, j prefers $A_N(\tilde{x})$ to $B_N(\tilde{x})$ at any level of wealth.

For $N = 2$, $i \trianglerighteq_2 j$ amounts to saying that if individual i accepts a gamble $\tilde{x}$ at some level of wealth, then j accepts the gamble at any level of wealth. It extends the concept "uniformly no less risk-averse" introduced by Aumann and Serrano (2008) by allowing $u^{(2)}$ to have arbitrary signs and $\tilde{x}$ to have arbitrary means.[5] For $N \geq 3$, $i \trianglerighteq_N j$ means that if individual i prefers to combine gamble $\tilde{x}$ with the "bad" existing lottery $A_{N-2}(\tilde{x})$ rather than with the "good" existing lottery $B_{N-2}(\tilde{x})$ at some level of wealth, then j makes the same decision at any level of wealth.

To quantify the intensity of risk apportionment, let us first define

$$\rho_{u,N}(w) = -\frac{u^{(N)}(w)}{u^{(N-1)}(w)}.$$

as the Nth-degree index of absolute risk aversion associated with u (Jindapon and Neilson 2007, Definition 2). Note that here we assume that $(-1)^N u^{(N-1)} > 0$ and that we do not require $\rho_{u,N} > 0$ in our analysis. The following result reveals an important link between $\trianglerighteq_N$ and $\rho_{u,N}$:

Theorem 1. *For $N \geq 2$, assume that $(-1)^N u_i^{(N-1)} > 0$ and $(-1)^N u_j^{(N-1)} > 0$. Then, $i \trianglerighteq_N j$ if and only if $\min_w \rho_{u_i,N}(w) \geq \max_w \rho_{u_j,N}(w)$.*

Theorem 1 translates the comparison of the intensity of risk apportionment into the comparison of the index of absolute risk aversion. Since $\trianglerighteq_N$ is a global concept applying to gambles of arbitrary size attached at any level of wealth, it is not surprising to see that $\trianglerighteq_N$ turns out to be a partial order characterized by a comparison of two global indexes $\min_w \rho_{u_i,N}(w)$ and $\max_w \rho_{u_j,N}(w)$.

Remark 1. Ross (1981) proposes an index stronger than second-degree absolute risk aversion, which is also generalized by Jindapon and Neilson (2007, Definition 1) to higher orders. According to this index, individual i is more Nth-degree Ross risk averse than individual j, if $\min_w \frac{u_i^{(N)}(w)}{u_j^{(N)}(w)} \geq \max_w \frac{u_i^{(1)}(w)}{u_j^{(1)}(w)}$. Our condition in Theorem 1 is neither sufficient nor necessary for Ross's greater risk aversion.[6]

[5] Aumann and Serrano (2008, p. 814) confine their concept to concave utility functions and gambles with positive means. In their discussion in Section IX.C, Aumann and Serrano assert that "Another avenue of research is to try to extend the approach to 'gambles' with negative expected value, which would apply to risk lovers. This could shed light on gambling behavior, for instance, and on high-risk 'venture capital'." Our study offers a formal extension.

[6] For $N = 2$, $u_i(w) = -e^{-aw}$ and $u_j(w) = -e^{-bw}$ with $a > b > 0$ satisfy our condition but not Ross's. Conversely, $u_i(w) = w - e^{-w}$ and $u_j(w) = w - be^{-w}$ with $0 < b < 1$ satisfy Ross's condition strictly but not ours. Counterexamples for $N \geq 3$ can be constructed in a similar spirit.

Theorem 1 serves our purpose in classifying individuals under the premise that $(-1)^N u^{(N-1)} > 0$. Consider a benchmark individual whose maximum Nth-degree index of absolute risk aversion equals $1-1/c$, with $c \in (0,1)$. Then, the class of individuals who exhibit uniformly no less intensity of risk apportionment of order N than the benchmark individual can be equivalently characterized by the class of utility functions that satisfy

$$\rho_{u,N} \geq (1-1/c).$$

3. $(N+c)$th-Degree Stochastic Dominance

Let $N \in \mathbb{N}$ and $c \in (0,1)$. Inspired by Theorem 1, our goal is to identify the dominance criteria associated with the following utility class

$$\mathscr{U}_{N,c} = \left\{ u \,\middle|\, (-1)^{n+1} u^{(n)} > 0,\ n = 1, \cdots, N,\ \rho_{u,N+1} \geq (1-1/c) \right\}.$$

$(N+c)$th-degree stochastic dominance is formally defined as follows:

Definition 3. Let $N \in \mathbb{N}$ and $c \in (0,1)$. Gamble $\tilde{x}$ dominates gamble $\tilde{y}$ by $(N+c)$**th-degree stochastic dominance**, written as $\tilde{x}$ (N+c)SD $\tilde{y}$, if $Eu(\tilde{x}) \geq Eu(\tilde{y})$ for all $u \in \mathscr{U}_{N,c}$.

The constraint on the lower bound of the $(N+1)$th-degree index of absolute risk aversion provides a natural interpolation of integer-degree stochastic dominance rules. If c goes to zero, the constraint on the $(N+1)$th-degree index of absolute risk aversion automatically holds and therefore the corresponding distribution ranking criterion approaches NSD. If c goes to 1, then the class of individuals exhibits mixed-risk aversion up to degree $N+1$ and therefore the decision criterion approaches (N+1)SD. Thus, (N+c)SD lies between NSD and (N+1)SD.

3.1. Distribution Conditions

To characterize (N+c)SD in terms of distribution conditions, assume that all gambles considered have support contained within the interval $[a,b]$. We allow for $a = -\infty$ or $b = \infty$, or both. For a cumulative distribution function (cdf) F, define

$$F^{(0)}(x) = F(x), \quad F^{(n)}(x) = \int_a^x F^{(n-1)}(t)dt \ \text{ for } n \geq 1.$$

The following theorem provides the distribution conditions for (N+c)SD:

Theorem 2. *Denote the cdfs of $\tilde{x}$ and $\tilde{y}$ by F and G and let $h_c(x) = e^{(1/c-1)x}$. For $N \in \mathbb{N}$ and $c \in (0,1)$, $\tilde{x}$ (N+c)SD $\tilde{y}$ if and only if*

$$G^{(n)}(b) \geq F^{(n)}(b),\ n = 1, \cdots, N-1,\ \int_a^x \left[G^{(N-1)}(t) - F^{(N-1)}(t)\right] dh_c(t) \geq 0,\ \forall x.$$

In the above, the condition $\int_a^x \left[G^{(N-1)}(t) - F^{(N-1)}(t)\right] dh_c(t) \geq 0$ approaches $G^{(N-1)}(x) \geq F^{(N-1)}(x)$ as $c \to 0$, and $G^{(N)}(x) \geq F^{(N)}(x)$ as $c \to 1$, which correspond to the condition for NSD and (N+1)SD (Eeckhoudt et al. 2009b, p. 96), respectively.[7]

It is obvious that $\mathscr{U}_{N,0} \supset \mathscr{U}_{N,c} \supset \mathscr{U}_{N+1,0}$ for $N \in \mathbb{N}$ and $c \in (0,1)$, following which we obtain

$$\text{NSD} \Longrightarrow \text{(N+c)SD} \Longrightarrow \text{(N+1)SD}.$$

Similarly, since $\mathscr{U}_{N,c_1} \supset \mathscr{U}_{N,c_2}$ if $0 < c_1 < c_2 < 1$, we also have

$$\text{(N+c}_1\text{)SD} \Longrightarrow \text{(N+c}_2\text{)SD}$$

As a result, by interpolating integer-degree stochastic dominance relations in our way, lower degree dominance always implies higher degree dominance.

3.2. Risk Apportionment via (N+c)SD

Let $N, M \in \mathbb{N}$. Eeckhoudt et al. (2009b) reveal that an individual exhibiting an (N+M)SD preference will allocate the state-contingent lotteries in such a way so as not to group the two "bad" lotteries in the same state, which generalizes the concept of risk apportionment. They show that

$$\tilde{x}_N \text{ NSD } \tilde{y}_N,\ \tilde{x}_M \text{ MSD } \tilde{y}_M \Longrightarrow [\tilde{x}_N + \tilde{y}_M; \tilde{y}_N + \tilde{x}_M] \text{ (N+M)SD } [\tilde{x}_N + \tilde{x}_M; \tilde{y}_N + \tilde{y}_M].$$

The following result generalizes Eeckhoudt et al. (2009b)'s finding to our fractional degree stochastic dominance relations:

Theorem 3. *For $c \in (0,1)$, if $\tilde{x}_{N+c}$ (N+c)SD $\tilde{y}_{N+c}$ and $\tilde{x}_M$ MSD $\tilde{y}_M$, then*

$$[\tilde{x}_{N+c} + \tilde{y}_M; \tilde{y}_{N+c} + \tilde{x}_M] \text{ (N+M+c)SD } [\tilde{x}_{N+c} + \tilde{x}_M; \tilde{y}_{N+c} + \tilde{y}_M].$$

Theorem 3 offers a way to generate the (N+M+c)SD relation by combining (N+c)SD with MSD. For example, let $\tilde{x}$ be a non-degenerate gamble and $\bar{x}_c = h_c^{-1}(Eh_c(\tilde{x}))$, which dominates $\tilde{x}$ by (1+c)SD. Taking N=1, we obtain that for any $k > 0$ and any pure risk $\tilde{\varepsilon}$:

M=1: $[\bar{x}_c - k; \tilde{x}]$ (2+c)SD $[\bar{x}_c; \tilde{x} - k]$;

M=2: $[\bar{x}_c + \tilde{\varepsilon}; \tilde{x}]$ (3+c)SD $[\bar{x}_c; \tilde{x} + \tilde{\varepsilon}]$;

M$\geq$ 3: $[\bar{x}_c + A_M(\tilde{\varepsilon}), \tilde{x} + B_M(\tilde{\varepsilon})]$ (M+1+c)SD $[\bar{x}_c + B_M(\tilde{\varepsilon}), \tilde{x} + A_M(\tilde{\varepsilon})]$.

In higher-order risk effects, precautionary effects capture the demand for saving when an uninsurable zero-mean risk is added to future wealth (Kimball 1990), and tempering effects (Kimball

[7] This is a corollary of the following result that can be verified by virtue of the L' Hôpital rule: For any continuous function $A(x)$ with $A(a) = 0$, $\lim_{c \to 1} \frac{c}{1-c} \int_a^x A(t) dh_c(t) = \int_a^x A(t) dt$ and $\lim_{c \to 0} \frac{1}{h_c(x)} \int_a^x A(t) dh_c(t) = A(x)$.

1991) point to the demand to reduce exposure to another independent risk when an unavoidable risk is introduced. Eeckhoudt et al. (2009b) reveal the parallel between these effects and risk apportionment. They interpret prudence as a type of preference for the disaggregation of a sure loss and an SSD deterioration for individuals within $\mathscr{U}_{3,0}$, and interpret temperance as a type of preference for the disaggregation of two independent SSD deteriorations for individuals within $\mathscr{U}_{4,0}$. By virtue of Theorem 3, we can generalize prudence as a type of preference for the disaggregation of a sure loss and a (1+c)SD deterioration for individuals within $\mathscr{U}_{2,c}$ (see Section 6.1 for a formal treatment). We can also generalize temperance as a type of preference for the disaggregation of an SSD deterioration and a (1+c)SD deterioration for individuals within $\mathscr{U}_{3,c}$.

A further question is how to combine two fractional degree stochastic dominance rules, say, (N+c_1)SD and (M+c_2)SD, $c_1, c_2 \in (0,1)$. When combining (N+c_1)SD and (M+c_2)SD, one method is to replace one relation with a higher integer-degree SD and then apply Theorem 3. In such a way, we will obtain (N+M+1+c_1)SD or (N+M+1+c_2)SD. The following theorem tightens the resulting relation to (N+M+1+c)SD with $c < \min\{c_1, c_2\}$.

Theorem 4. *For $c_1, c_2 \in (0,1)$, if $\tilde{x}_{N+c_1}$ (N+c_1)SD $\tilde{y}_{N+c_1}$ and $\tilde{x}_{M+c_2}$ (M+c_2)SD $\tilde{y}_{M+c_2}$, then*

$$[\tilde{x}_{N+c_1} + \tilde{y}_{M+c_2}; \tilde{y}_{N+c_1} + \tilde{x}_{M+c_2}] \ \text{(N+M+1+c)SD} \ [\tilde{x}_{N+c_1} + \tilde{x}_{M+c_2}; \tilde{y}_{N+c_1} + \tilde{y}_{M+c_2}],$$

where c satisfies $\frac{1}{c} + 1 = \frac{1}{c_1} + \frac{1}{c_2}$.

Theorem 4 shows that, for example, combining 1.8SD and 2.8SD yields 4.67SD. If either c_1 or c_2 approaches zero, then c approaches zero as well. If c_i approaches 1, then c approaches c_j, $i, j = 1, 2$.

3.3. $(N+c)$th-Degree Increase in Risk

In defining (N+c)SD, we impose the requirement that individuals exhibit mixed-risk aversion up to degree N. The following variant removes the requirements on the signs of lower-order derivatives. Let

$$\mathscr{U}'_{N,c} = \left\{ u \,\middle|\, (1-c)(-1)^{N+1} u^{(N)} + c(-1)^{N+2} u^{(N+1)} > 0 \right\}.$$

Definition 4. Let $N \in \mathbb{N}$ and $c \in (0,1)$. Gamble $\tilde{y}$ is an $(N+c)$**th-degree increase in risk** relative to gamble $\tilde{x}$, written as $\tilde{y}$ (N+c)IR $\tilde{x}$, if $Eu(\tilde{x}) \geq Eu(\tilde{y})$ for all $u \in \mathscr{U}'_{N,c}$.

Compared with (N+c)SD, (N+c)IR puts less restrictions on utility functions and hence results in stricter restrictions on distributions. Based on the set inclusion $\mathscr{U}'_{N,c} \subset \mathscr{U}_{N,c}$, we derive the distribution condition for (N+c)IR as

$$G^{(n)}(b) = F^{(n)}(b), \ n = 1, \cdots, N-1, \ \int_a^b \left[G^{(N-1)}(t) - F^{(N-1)}(t)\right] dh_c(t) = 0, \quad \text{and}$$
$$\int_a^x \left[G^{(N-1)}(t) - F^{(N-1)}(t)\right] dh_c(t) \geq 0, \ \forall x.$$

Accordingly, $\tilde{y}$ (N+c)IR $\tilde{x}$ implies that $\tilde{x}$ and $\tilde{y}$ have the same first $N-1$ moments. As $c \to 0$ and $c \to 1$, (N+c)IR approaches NIR and (N+1)IR defined by Ekern (1980), respectively.[8] Unlike (N+c)SD, (N+c)IR does not exhibit an inclusion relation across integer degrees. However, the result in Theorem 3 is extendable to (N+c)IR.

Corollary 1. *For $c \in (0,1)$, if $\tilde{y}_{N+c}$ (N+c)IR $\tilde{x}$ and $\tilde{y}_M$ MIR $\tilde{x}_M$, then*

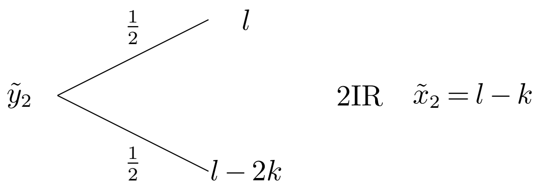
$$[\tilde{x}_{N+c} + \tilde{x}_M; \tilde{y}_{N+c} + \tilde{y}_M] \ \textit{(N+M+c)IR} \ [\tilde{x}_{N+c} + \tilde{y}_M; \tilde{y}_{N+c} + \tilde{x}_M] .$$

The following example illustrates how to construct (2+c)IR by combining (1+c)IR with FIR.

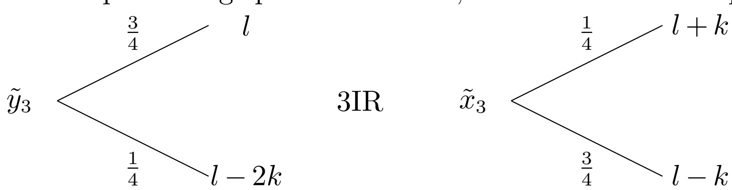
Example 1 (Between risk and downside risk). Given $c \in (0,1)$ and $k > 0$, let $\tilde{x}_1 = l$, $\tilde{y}_1 = l - k$, $\tilde{x}_{1+c} = 0$, and $\tilde{y}_{1+c}$ be a Bernoulli gamble that takes k with probability $1/(1 + h_c(k))$ and $-k$ with probability $h_c(k)/(1 + h_c(k))$. Then, $\tilde{y}_1$ FIR $\tilde{x}_1$, $\tilde{y}_{1+c}$ (1+c)IR $\tilde{x}_{1+c}$, and

$$[\tilde{x}_{1+c} + \tilde{x}_1; \tilde{y}_{1+c} + \tilde{y}_1] \ \text{(2+c)IR} \ [\tilde{x}_{1+c} + \tilde{y}_1; \tilde{y}_{1+c} + \tilde{x}_1] .$$

As $c \to 0$, the above relation approaches



which reproduces the mean-preserving spread. As $c \to 1$, the above relation approaches



which reproduces the mean-variance-preserving transformation that is used by Menezes et al. (1980) to illustrate the concept of downside risk.

4. Alternative Characterizations of (1+c)SD

Since FSD and SSD are the most popular rules, this section offers alternative characterizations of (1+c)SD to gain additional insights into the choice behavior of individuals with a limited degree of risk lovingness.

[8] Rothschild and Stiglitz (1970) describe an *increase in risk* as a mean-preserving spread, which is disliked by all individuals with $u^{(2)} < 0$. Menezes et al. (1980) define an *increase in downside risk* as a mean-variance-preserving transformation, which is disliked by all individuals with $u^{(3)} > 0$. Ekern (1980) unifies these concepts under the terminology of an *Nth-degree increase in risk*, which is referred to as a change in risks disliked by all individuals with $(-1)^{N+1}u^{(N)} > 0$. As pointed out by Ekern (1980), G has more Nth-degree risk than F, if and only if F NSD G and meanwhile F and G have the same first $N-1$ moments.

For $N=1$, $G^{(0)}(b)=F^{(0)}(b)=1$, and the distribution condition for (1+c)SD collapses to

$$\int_a^x [G(t)-F(t)]\,dh_c(t)\geq 0,\ \forall x.$$

As $\int_a^x F(t)dh_c(t)=\int_a^x [h_c(x)-h_c(t)]\,dF(t)$ is the expectation of the excess loss measured by $h_c(x)-h_c(t)$, the above condition on (1+c)SD amounts to saying that the expected excess loss for F is everywhere not bigger than G.

Remark 2. The condition for (1+c)SD has been seen in Meyer (1977a,b), in which it was referred to as "F stochastically dominates G in the second degree with respect to h_c." However, Meyer (1977b)'s purpose is to propose an SSD rule for individuals with a lower bound on their index of absolute risk aversion. By contrast, our purpose is to provide an interpolation between FSD and SSD and accordingly we concentrate on $c<1$.

4.1. Probability Transfer

Recall that FSD captures the aversion to a mean-decreasing deterioration and SSD captures the aversion to a mean-preserving spread. As an ordering falling in between them, (1+c)SD captures the aversion to a kind of *mean-decreasing spread*. This kind of spread can be characterized explicitly in terms of a probability transfer, the basic forms of which are provided as follows:

Definition 5. Consider two discrete gambles with cdfs F and G.

(a) F is obtained from G via an *increasing transfer* if there exist $x_1<x_2$ and $\eta>0$ such that

$$G(x)-F(x)=\begin{cases}0, & \text{if } x<x_1,\\ \eta, & \text{if } x_1\leq x<x_2,\\ 0, & \text{if } x\geq x_2.\end{cases}$$

(b) F is obtained from G via a *c-transfer* if there exist $x_1<x_2<x_3<x_4$ and $\eta_1,\eta_2>0$ with $\eta_1\,[h_c(x_2)-h_c(x_1)]=\eta_2\,[h_c(x_4)-h_c(x_3)]$ such that

$$G(x)-F(x)=\begin{cases}0, & \text{if } x<x_1,\\ \eta_1, & \text{if } x_1\leq x<x_2,\\ 0, & \text{if } x_2\leq x<x_3,\\ -\eta_2, & \text{if } x_3\leq x<x_4,\\ 0, & \text{if } x\geq x_4.\end{cases}$$

When the gamble undergoes an increasing transfer in distribution, its cdf becomes strictly smaller in some range and thus its mean becomes larger. When the gamble undergoes a c-transfer for some $c\in(0,1)$, as shown in Figure 1, there must be

$$\eta_1(x_2-x_1)>\eta_2(x_4-x_3).$$

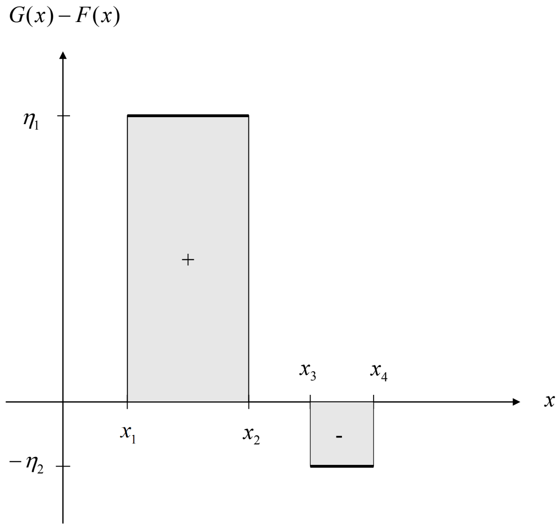


Figure 1 **An illustration of c-transfer [$c=1/2$, $(x_1,x_2,x_3,x_4)=(1,3,4,5)$, $\eta_1=5.4$ and $\eta_2=1$].**

That is, the area in which $F<G$ (the "+" area in Figure 1) is larger than the area in which $F>G$ (the "-" area in Figure 1). Therefore, a c-transfer also results in an increase in the mean. The two transfers, when working together, make F a *mean-increasing contraction* of G, or equivalently, G a *mean-decreasing spread* of F. As $c\to 0$, the condition for the c-transfer converges to $\eta_1>0$, $\eta_2=0$, which corresponds to an increasing transfer for FSD. As $c\to 1$, the condition for the c-transfer converges to $\eta_1(x_2-x_1)=\eta_2(x_4-x_3)$, which corresponds to a mean-preserving contraction for SSD.

Theorem 5. *Given $c\in(0,1)$, $\tilde{x}$ (1+c)SD $\tilde{y}$, if and only if there exist two discrete sequences $\tilde{x}_m$ and $\tilde{y}_m$ that converge in distribution to $\tilde{x}$ and $\tilde{y}$, respectively, such that for all $m\in\mathbb{N}$, the distribution of $\tilde{x}_m$ can be obtained from the distribution of $\tilde{y}_m$ via a finite sequence of c-transfers and increasing transfers.*[9] *Moreover, $E[\tilde{x}]\geq E[\tilde{y}]$, with equality holding if and only if $\tilde{x}$ and $\tilde{y}$ have identical distributions.*

Theorem 5 clarifies that under $c<1$, except for the trivial case where $\tilde{x}=\tilde{y}$ in distribution, a necessary condition for $\tilde{x}$ (1+c)SD $\tilde{y}$ is $E[\tilde{x}]>E[\tilde{y}]$. This necessity captures the intuition that by allowing for individuals to be risk-loving to some extent, a contraction in payoffs is preferable for all individuals only if it yields a strictly higher expected return. Without a positive compensation, some individuals within $\mathscr{U}_{1,c}$ will dislike the contraction.

Corollary 2. *Let $\tilde{x}_1$ and $\tilde{x}_2$ be non-degenerate independently and identically distributed gambles. Given $c\in[0,1]$, $\frac{1}{2}\tilde{x}_1+\frac{1}{2}\tilde{x}_2$ (1+c)SD $\tilde{x}_1$ if and only if $c=1$.*

[9] Müller et al. (2016) employ a family of γ-transfers to characterize their fractional-degree SD. Our Theorem 5 is in the same spirit as theirs but with a distinct family of transfers.

Corollary 2 states that the long-standing recommendation for diversification (Samuelson 1967) works parameter-freely only for risk-averse investors. When the two assets are independently and identically distributed, diversification induces a payoff contraction but never yields a positive compensation. Thus, compared to investing entirely in a single asset, diversification will become less appealing to some investors within $\mathscr{U}_{1,c}$ as long as $c < 1$.

4.2. Single-Crossing Distributions

The condition for (1+c)SD is particularly simple if distributions are single-crossing.

THEOREM 6. *Suppose there exists a single crossing point $\hat{x} \in (a,b)$ such that $F(x) < (>) G(x)$ for $x < (>) \hat{x}$. $\tilde{x}$ (1+c)SD $\tilde{y}$ if and only if $E[h_c(\tilde{x})] \geq E[h_c(\tilde{y})]$.*

The single-crossing property is met automatically if the distributions come from the same *location-scale family*, namely, if

$$F(x) = \Psi\left(\frac{x - \mu_F}{\sigma_F}\right) \qquad \text{and} \qquad G(x) = \Psi\left(\frac{x - \mu_G}{\sigma_G}\right),$$

where $\sigma_F, \sigma_G > 0$, and Ψ is the cdf of a random variable $\tilde{z}$ with a mean of zero and a standard deviation of one. In such a pair of gambles, $\tilde{x}$ ($\tilde{y}$) has mean μ_F (μ_G) and variance σ_F^2 (σ_G^2). The cdfs F and G cross exactly once at $\hat{x} = (\mu_G \sigma_F - \mu_F \sigma_G)/(\sigma_F - \sigma_G)$ when $\sigma_F \neq \sigma_G$. Thanks to Theorem 6, given $\sigma_F \leq \sigma_G$, $\tilde{x}$ (1+c)SD $\tilde{y}$ if and only if

$$\mu_F - \mu_G \geq h_c^{-1}\left(\frac{E[h_c(\sigma_G \tilde{z})]}{E[h_c(\sigma_F \tilde{z})]}\right).$$

EXAMPLE 2 ((1+C)SD FOR NORMAL DISTRIBUTIONS). A special location-scale family that is commonly used is the normal distribution, for which given $\sigma_F \leq \sigma_G$, $\tilde{x}$ (1+c)SD $\tilde{y}$ if and only if

$$\mu_F - \mu_G \geq \frac{1}{2}\left(\frac{1}{c} - 1\right)(\sigma_G^2 - \sigma_F^2).$$

For $\tilde{x}$ to dominate, the compensation in the mean measured as $\mu_F - \mu_G$ is decreasing in c. It approaches zero as $c \to 1$, but approaches infinity as $c \to 0$.

Another special location-scale family relates to the distributions generated by an affine transformation. The following corollary is straightforward.

COROLLARY 3. *Let $\beta \in [0,1]$ and $\tilde{x}_{\alpha,\beta} = \alpha + \beta \tilde{x}$. $\tilde{x}_{\alpha,\beta}$ (1+c)SD $\tilde{x}$ if and only if*

$$\alpha \geq h_c^{-1}\left(\frac{Eh_c(\tilde{x})}{Eh_c(\beta \tilde{x})}\right).$$

In Corollary 3, as $c \to 0$, we obtain that the equivalent condition for an FSD improvement is $\alpha \geq (1-\beta)\text{ess-sup}[\tilde{x}]$; as $c \to 1$, we reproduce the result first offered by Hadar and Russell (1971) where the equivalent condition for an SSD improvement is $\alpha \geq (1-\beta)E[\tilde{x}]$.[10] The case $\beta = 0$ indicates that the minimum sure gain dominating $\tilde{x}$ by (1+c)SD is

$$\bar{x}_c = h_c^{-1}\left(Eh_c(\tilde{x})\right),$$

which approaches ess-sup$[\tilde{x}]$ as $c \to 0$ and approaches $E[\tilde{x}]$ as $c \to 1$. For $\tilde{x}$ subject to a normal distribution $N(\mu, \sigma^2)$, $\bar{x}_c = \mu + \sigma^2\left(1/c - 1\right)/2$.

Corollary 3 specifies the condition for a portfolio with a riskless asset to be more appealing relative to the single risky asset. Indeed, $\tilde{x}_{(1-\beta)r,\beta} = (1-\beta)r + \beta\tilde{x}$, $\beta \in (0,1)$, describes the payoff of a portfolio consisting of a riskless asset and a risky asset (Levy and Kroll 1976), where r is the net return of the riskless asset, $\tilde{x}$ is the net return of the risky asset, and β denotes the proportion of the investor's wealth invested in the risky asset. For $\tilde{x}_{(1-\beta)r,\beta}$ to be preferable to $\tilde{x}$ for all investors within $\mathscr{U}_{1,c}$, the necessary and sufficient condition is $r \geq \frac{1}{1-\beta} h_c^{-1}\left(\frac{Eh_c(\tilde{x})}{Eh_c(\beta\tilde{x})}\right)$.

Example 3 (Dominance of portfolio). For $\tilde{x}$ subject to $N(\mu, \sigma^2)$ and $\beta \in (0,1)$, $\tilde{x}_{(1-\beta)r,\beta}$ (1+c)SD $\tilde{x}$ if and only if

$$r \geq \mu + \frac{1}{2}\left(\frac{1}{c} - 1\right)(1+\beta)\sigma^2.$$

4.3. Comparison with Related Literature

The idea of establishing a continuum of stochastic dominance relations has appeared before in the literature. Fishburn (1976, 1980) uses fractional integration to define the relations. His method, however, lacks a straight economic interpretation on the parameter used for interpolation, and his distribution conditions are hard to check even for the simplest Bernoulli gambles (Müller et al. 2016, p. 6).

Müller et al. (2016) define $\tilde{x}$ (1+γ)SD $\tilde{y}$ with $\gamma \in [0,1]$ as $E[u(\tilde{x})] \geq E[u(\tilde{y})]$ for all u satisfying $0 \leq \gamma u'(y) \leq u'(x)$ for all $x \leq y$. Although we both aim at formulating a sense of mean-decreasing spread, our (1+c)SD differs from their (1+γ)SD in three respects.

First, the utility classes are different. We require the risk aversion to be bounded from below while Müller et al. (2016) introduce a bound on how much marginal utility decreases as wealth decreases. Note that Müller et al. (2016)'s condition can also be interpreted as a bound on the index of greediness introduced by Chateauneuf et al. (2005). Compared with the index of greediness, the index of absolute risk aversion is better understood in the literature and can be linked to many well-defined forms of economic behavior such as self-protection (Jindapon and Neilson 2007) and the substitution of risk increase (Liu and Meyer 2013). More importantly, the economic justification

[10] Applying the L'Hôpital rule to $h_c^{-1}(Eh_c(\tilde{x}))$, we obtain $\lim_{c\to 0} h_c^{-1}(Eh_c(\tilde{x})) =$ ess-sup$[\tilde{x}]$ and $\lim_{c\to 1} h_c^{-1}(Eh_c(\tilde{x})) = E[\tilde{x}]$. The limiting results then follow straightforwardly from $h_c^{-1}\left(\frac{Eh_c(\tilde{x})}{Eh_c(\beta\tilde{x})}\right) = h_c^{-1}(Eh_c(\tilde{x})) - h_c^{-1}(Eh_c(\beta\tilde{x}))$.

for the index of absolute risk aversion can be extended straightforwardly to higher orders. In comparison, extending Müller et al. (2016)'s approach to higher orders by limiting the ratio of Nth-order derivatives on utility functions makes less sense.

Second, the differentiability condition in Müller et al. (2016)'s utility class is not critical, and hence utility functions with kinks can be accommodated by (1+γ)SD. By contrast, differentiability is crucial for us so that our utility class cannot allow for kinks. However, given that any continuous utility function with countable kinks can be approximated by smooth utility functions, our utility class covers the approximations of preferences with kinks. Moreover, for individuals who are endowed with an initial smooth background risk $\tilde{\varepsilon}$, the indirect utility function $v(w)=Eu(w+\tilde{\varepsilon})$ is always smooth, as long as $u(w)$ is continuous.[11] Therefore, our utility class does not lose any generality to the analysis of the economic behavior for such individuals.

Third, the probability transfers are different. For (1+γ)SD, the ratio of positive to negative probability mass transfers is always constant. However, for (1+c)SD, this ratio changes with the location of the mass transfers. As a result, the distribution conditions are different. Sometimes, the distribution condition for (1+c)SD is more tractable.

Example 4 ((1+γ)SD for normal distributions). For normal distributions, the distribution condition for (1+γ)SD is

$$\sigma_F \leq \sigma_G \quad \text{ and } \quad \mu_F - \mu_G \geq \left(\frac{1}{\gamma} - 1\right) \int_{\left(\frac{\mu_G\sigma_F - \mu_F\sigma_G}{\sigma_F - \sigma_G}\right)}^{\infty} \left(\Phi\left(\frac{x-\mu_F}{\sigma_F}\right) - \Phi\left(\frac{x-\mu_G}{\sigma_G}\right)\right) dx,$$

where Φ is the standard normal cfd. By comparison, the distribution condition for (1+c)SD offered in Example 2 is much simpler.

5. Extensions on New Notions of Stochastic Dominance

Up to now, we have established the dominance criteria for the utility class identified by a lower bound on the $(N+1)$th-degree absolute risk aversion. Our methodology, however, is not limited to one specific way of classifying individuals. By replacing the *negative lower bound* on risk aversion with a *positive upper bound*, or replacing the *absolute risk aversion* with *relative risk aversion*, we can adopt our methodology to obtain two other new notions of stochastic dominance.

5.1. $(1+c)$th-Degree Convex Stochastic Dominance

The NSD with $(-1)^{n+1}u^{(n)}>0$, $n=1,\cdots,N$, is usually termed "concave stochastic dominance" in the literature (Denuit et al. 2013). As a counterpart, the "convex stochastic dominance" characterizes the common criterion of risk choices for risk lovers with $(-1)^{n+1}u^{(n)}<0$ (Levy and Wiener

[11] Denote by $f(\varepsilon)$ the density of $\tilde{\varepsilon}$ that is infinitely differentiable, and assume that $u(w)$ is continuous. Then, $v(w)$ is infinitely differentiable with $v^{(n)}(w)=(-1)^n\int u(y)f^{(n)}(y-w)dy$, even if $u(w)$ has some kinks.

1998, Post and Levy 2005). Imposing a positive upper bound on the $(N+1)$th-degree index of absolute risk aversion allows us to establish an interpolation of convex stochastic dominance. To simplify matters, we elaborate on the new stochastic dominance rule with a degree between $N=1$ and $N=2$. Extensions to higher orders are straightforward.

To distinguish such stochastic dominance from SSD, we denote the second-degree convex stochastic dominance for $N=2$ by SSD*. Define a set of individuals as

$$\mathscr{U}^*_{1,c} = \left\{ u \,\middle|\, u^{(1)} > 0,\ \rho_{u,2} \le (1/c-1) \right\}.$$

Individuals in this set are non-satiable. They could be risk-averse or risk-loving but they are not too risk-averse. If c goes to zero, the constraint on the risk aversion automatically holds and therefore the corresponding distribution ranking criterion approaches FSD. If c goes to 1, then the class of individuals exhibits risk lovingness and therefore the decision criterion approaches SSD*.

Definition 6. Gamble $\tilde{x}$ dominates gamble $\tilde{y}$ by $(1+c)$**th-degree convex stochastic dominance**, written as $\tilde{x}$ (1+c)SD* $\tilde{y}$, if $Eu(\tilde{x}) \ge Eu(\tilde{y})$ for all $u \in \mathscr{U}^*_{1,c}$.

Theorem 7. *Denote the cdfs of $\tilde{x}$ and $\tilde{y}$ by F and G and let $h^*_c(x) = -e^{(1-1/c)x}$. For $c \in (0,1)$, $\tilde{x}$ (1+c)SD* $\tilde{y}$ if and only if*

$$\int_x^b [G(t) - F(t)]\, dh^*_c(t) \ge 0,\ \forall x,$$

The (1+c)SD* rule is intimately related to (1+c)SD. Noticing that $u(w)$ is convex increasing if and only if $u^*(w) = -u(-w)$ is concave increasing, we have that $u \in \mathscr{U}^*_{1,c}$ if and only if $u^* \in \mathscr{U}_{1,c}$. Then, $\tilde{x}$ (1+c)SD* $\tilde{y}$ is equivalent to $-\tilde{y}$ (1+c)SD $-\tilde{x}$. Since (1+c)SD captures the aversion to a kind of mean-decreasing spread, it is easily seen that (1+c)SD* captures the aversion to a kind of mean-decreasing contraction.

A more generalized extension would be imposing both a positive upper bound and negative lower bound on the absolute risk aversion, which includes concave and convex (1+c)SD as special cases. Formally, let $c_1, c_2 \in (0,1)$ and consider the utility class

$$\mathscr{U}^*_{1,c_1,c_2} = \left\{ u \,\middle|\, u^{(1)} > 0,\ (1-1/c_1) \le \rho_{u,2} \le (1/c_2 - 1) \right\}.$$

Then, we can define $(1+c_1, 1+c_2)$th-degree concave-convex stochastic dominance as a preference $Eu(\tilde{x}) \ge Eu(\tilde{y})$ for all $u \in \mathscr{U}^*_{1,c_1,c_2}$. In the same spirit of Meyer (1977a), we can translate the corresponding distribution condition into checking whether the minimum of the following problem

$$\min_{u \in \mathscr{U}^*_{1,c_1,c_2}} \int_a^b [G(x) - F(x)] u^{(1)}(x) dx \quad \text{s.t.} \quad u^{(1)}(0) = 1$$

is nonnegative. Although no closed-form expression is available in general for the minimum, the minimum can be calculated following a well-defined procedure attributed to Meyer (1977a).

5.2. (1+c)SD under Relative Risk Aversion

Since the seminal works of Arrow (1971) and Pratt (1964), both absolute risk aversion and relative risk aversion have been well applied to measure the intensity of individuals' risk aversion. Absolute risk aversion works for additive absolute returns, while relative risk aversion works for multiplicative relative returns (Schreiber 2014). Accordingly, a natural extension is to interpolate NSD with relative risk aversion. We illustrate this idea by focusing on $N=1$ and $N=2$. Extension to higher orders is achievable in principle through extending the concept of "uniformly no less relative risk-averse" to higher orders by using the notion of risk apportionment for multiplicative lotteries (Eeckhoudt et al. 2009a, Chiu et al. 2012).

Assuming that $u^{(1)}>0$, we use

$$\rho^r_{u,2}(w)=-\frac{wu^{(2)}(w)}{u^{(1)}(w)}$$

to denote the index of relative risk aversion associated with u (Arrow 1971). Parallel to Definition 2 and Theorem 1, the following definition and theorem provide support for the rationale of confining individuals to

$$\mathscr{U}^r_{1,c}=\left\{u\,\middle|\,u^{(1)}>0,\ \rho^r_{u,2}\geq(1-1/c)\right\}.$$

The gamble involved here is multiplicative as its payoff is measured in relative terms.

Definition 7. We say that individual i is **uniformly no less relative risk-averse** than individual j, written as $i\trianglerighteq^r_2 j$, if for any multiplicative gamble, whenever i accepts it at some level of wealth, j accepts it at any level of wealth.

Theorem 8. *Assume that $u_i^{(1)}>0$ and $u_j^{(1)}>0$. Then, $i\trianglerighteq^r_2 j$ if and only if $\min_w\rho^r_{u_i,2}(w)\geq\max_w\rho^r_{u_j,2}(w)$.*

Definition 8. Gamble $\tilde{x}$ dominates gamble $\tilde{y}$ by $(1+c)$**th-degree stochastic dominance under relative risk aversion**, written as $\tilde{x}$ (1+c)SD^r $\tilde{y}$, if $Eu(\tilde{x})\geq Eu(\tilde{y})$ for all $u\in\mathscr{U}^r_{1,c}$.

Theorem 9. *Denote the cdfs of $\tilde{x}$ and $\tilde{y}$ by F and G and let $h^r_c(x)=cx^{1/c}$. For $c\in(0,1)$, $\tilde{x}$ (1+c)SD^r $\tilde{y}$ if and only if*

$$\int_a^x[G(t)-F(t)]\,dh^r_c(t)\geq0,\ \forall x,$$

Like (1+c)SD, (1+c)SD^r also provides a continuum between FSD and SSD. As the only distinction between (1+c)SD and (1+c)SD^r is the weighting function in the integrand, Theorem 5 and Theorem 6 in Section 4 can be made valid for (1+c)SD^r if we replace h_c therein with h^r_c.

Finally, an important question could be raised as to when we should use (1+c)SD and when we should use (1+c)SD^r. Generally speaking, (1+c)SD falls in the framework where risks are in absolute terms and additive, while (1+c)SD^r fits the framework where risks are in relative terms and multiplicative. A more precise distinction between them is that (1+c)SD satisfies translation invariance but not scaling invariance, whereas (1+c)SD^r satisfies scaling invariance but not translation invariance.[12] Thus, whether we should use (1+c)SD or (1+c)SD^r to link FSD and SSD will be up to whether risks are measured in an absolute or relative sense, and whether translation invariance or scaling invariance should be taken into consideration.

6. Applications

In this section, we illustrate that (N+c)SD can be readily applied in broad areas of economics. Specifically, the existing literature has carefully studied the impact on the decision with respect to a change in risk in terms of NSD. Our findings can be used to analyze the cases where the change in risk is under the concept of (N+c)SD.

6.1. Optimal Precautionary Saving

Within a two-period setting, Kimball (1990) documented that prudence is the necessary and sufficient condition for an increase in the optimal saving level in the presence of an independent zero-mean background risk in the second period. Recently, Eeckhoudt and Schlesinger (2008) and Nocetti (2016) examined the effect of a deterioration in risk in terms of NSD in the background risk on the saving decision. We extend the analysis to risk changes in line with (N+c)SD.

Formally, assume that an individual with a certain income w lives in two periods. In the first period, the individual decides to save s in the form of a risk-free asset which will generate a positive return r in the second period. Let u_0 and u_1 denote the utility functions in the first and second periods, respectively. The optimal saving with a background risk $\tilde{\varepsilon}_i$, $i=1,2$, in the second period corresponds to the following problem

$$s_i^* = \arg\max_s EU(s,\tilde{\varepsilon}_i), \text{ where } U(s,\tilde{\varepsilon}_i) = u_0(w-s) + u_1(w+s(1+r)+\tilde{\varepsilon}_i).$$

Following Nocetti (2016) and Wang and Li (2015), we focus on a subset of individuals within $\mathscr{U}_{N,c}$, whose expected utility is *strictly* decreasing in the (N+c)SD deterioration.

Definition 9. We say that $u \in \mathscr{U}^\circ_{N,c}$, if for any pair $(\tilde{x},\tilde{y})$ such that $\tilde{x}$ (N+c)SD $\tilde{y}$ and $\tilde{x} \neq \tilde{y}$, we always have $Eu(\tilde{x}) > Eu(\tilde{y})$.

In what follows, we show that the optima are chosen in alignment with the preference for risk apportionment. To see it, we express the optima conditions as $EU(s_1^*,\tilde{\varepsilon}_1) \geq EU(s_2^*,\tilde{\varepsilon}_1)$ and $EU(s_2^*,\tilde{\varepsilon}_2) \geq EU(s_1^*,\tilde{\varepsilon}_2)$. Putting these two inequalities together yields

$$\begin{aligned}&\frac{1}{2}Eu_1(w+s_1^*(1+r)+\tilde{\varepsilon}_1) + \frac{1}{2}Eu_1(w+s_2^*(1+r)+\tilde{\varepsilon}_2)\\ \geq &\frac{1}{2}Eu_1(w+s_2^*(1+r)+\tilde{\varepsilon}_1) + \frac{1}{2}Eu_1(w+s_1^*(1+r)+\tilde{\varepsilon}_2).\end{aligned}$$

[12] Formally, for gambles $\tilde{x}$ and $\tilde{y}$, if $\tilde{x}$ (1+c)SD $\tilde{y}$, then $(b+\tilde{x})$ (1+c)SD $(b+\tilde{y})$, where b is a constant; if $\tilde{x}$ (1+c)SD^r $\tilde{y}$, then $a\tilde{x}$ (1+c)SD^r $a\tilde{y}$, where a is a positive constant.

Given $\tilde{\varepsilon}_1$ (N+c)SD $\tilde{\varepsilon}_2$, we claim that $s_1^* \leq s_2^*$ for all individuals with $u_1 \in \mathscr{U}_{N+1,c}^{\circ}$. Otherwise, we apply Theorem 3 together with the fact that $s_1^*(1+r)$ FSD $s_2^*(1+r)$ to conclude

$$[s_2^*(1+r)+\tilde{\varepsilon}_1; s_1^*(1+r)+\tilde{\varepsilon}_2] \text{ (N+1+c)SD } [s_1^*(1+r)+\tilde{\varepsilon}_1; s_2^*(1+r)+\tilde{\varepsilon}_2],$$

which leads to the opposite of the above inequality for $u_1 \in \mathscr{U}_{N+1,c}^{\circ}$. Noting that the above analysis does not hinge upon the first-order condition, the result remains valid even with multiple optima.[13]

Corollary 4. *Let $N \in \mathbb{N}$ and $c \in (0,1)$. If $\tilde{\varepsilon}_1$ (N+c)SD $\tilde{\varepsilon}_2$ and $\tilde{\varepsilon}_1 \neq \tilde{\varepsilon}_2$, then $s_2^* \geq s_1^*$ for all individuals with $u_1 \in \mathscr{U}_{N+1,c}^{\circ}$.*

6.2. Optimal Precautionary Effort

Here, we illustrate Theorem 3 in the context of precautionary efforts. The conditions for which the deterioration of an independent background risk increases effort (or prevention) under a two-period setting have been examined in Menegatti (2009), Eeckhoudt et al. (2012), Wang and Li (2015) and Nocetti (2016). Basically, they assume that the individual has a certain income w_0 in the first period. In the second period, her income is uncertain, and could be either w_1 or $w_1 - l$ with some $l > 0$. The individual can exert effort e in the first-period to increase the probability of a favorite outcome, w_1, in the second period. Let $p(e)$ be the probability of w_1 with $p^{(1)}(e) > 0$. Let u_0 and u_1 denote, respectively, the utility functions in the first and second periods. The optimal effort with an independent background risk $\tilde{\varepsilon}_i$, $i = 1,2$, corresponds to the following problem

$$e_i^* = \arg\max_e EU(e, \tilde{\varepsilon}_i), \text{ where } U(e,\tilde{\varepsilon}_i) = u_0(w_0 - e) + p(e)u_1(w_1 + \tilde{\varepsilon}_i) + (1-p(e))u_1(w_1 + \tilde{\varepsilon}_i - l).$$

Given $\tilde{\varepsilon}_1$(N+c)SD$\tilde{\varepsilon}_2$, we claim that $e_2^* \geq e_1^*$ for all individuals with $u_1 \in \mathscr{U}_{N+1,c}^{\circ}$. Otherwise, we write the optima conditions as $EU(e_1^*, \tilde{\varepsilon}_1) \geq EU(e_2^*, \tilde{\varepsilon}_1)$ and $EU(e_2^*, \tilde{\varepsilon}_2) \geq EU(e_1^*, \tilde{\varepsilon}_2)$, which can be further translated into

$$\frac{1}{2}Eu_1(w_1+\tilde{\varepsilon}_1) + \frac{1}{2}Eu_1(w_1+\tilde{\varepsilon}_2 - l) \geq \frac{1}{2}Eu_1(w_1+\tilde{\varepsilon}_2) + \frac{1}{2}Eu_1(w_1+\tilde{\varepsilon}_1 - l).$$

We apply Theorem 3 together with the fact that 0 FSD $-l$ to conclude

$$[\tilde{\varepsilon}_1 - l; \tilde{\varepsilon}_2] \text{ (N+1+c)SD } [\tilde{\varepsilon}_1; \tilde{\varepsilon}_2 - l],$$

[13] Alternatively, one can assume instead that $u_0^{(1)} > 0$, $u_0^{(2)} < 0$, and $u_1 \in \mathscr{U}_{N+1,c}$. Then, $EU(s, \tilde{\varepsilon}_i)$ is concave in s and the optimal saving decision is uniquely determined by the first-order condition $EU_s(s_i^*, \tilde{\varepsilon}_i) = 0$. Therefore, $s_2^* \geq s_1^*$ if and only if $EU_s(s_1^*, \tilde{\varepsilon}_2) \geq 0 = EU_s(s_1^*, \tilde{\varepsilon}_1)$, which after some algebraic manipulations turns out to be $E[v_1(w + s_1^*(1+r) + \tilde{\varepsilon}_1)] \geq E[v_1(w + s_1^*(1+r) + \tilde{\varepsilon}_2)]$, with $v_1 = -u_1^{(1)}$. Since $u_1 \in \mathscr{U}_{N+1,c}$ implies that $v_1 \in \mathscr{U}_{N,c}$, we come to the conclusion that for $u_1 \in \mathscr{U}_{N+1,c}$, $s_2^* \geq s_1^*$ if $\tilde{\varepsilon}_1$ (N+c)SD $\tilde{\varepsilon}_2$.

which leads to the opposite of the above inequality for $u_1 \in \mathscr{U}^{\circ}_{N+1,c}$. Again, the above analysis does not rely on the first-order condition, and hence the result holds true even with multiple optima.[14]

Corollary 5. *Let $N \in \mathbb{N}$, $c \in (0,1)$, and $p^{(1)} > 0$. If $\tilde{\varepsilon}_1$ (N+c)SD $\tilde{\varepsilon}_2$ and $\tilde{\varepsilon}_1 \neq \tilde{\varepsilon}_2$, then $e_2^* \geq e_1^*$ for all individuals with $u_1 \in \mathscr{U}^{\circ}_{N+1,c}$.*

6.3. Optimal Risky Investment

Expectation dependence established by Wright (1987) has been applied to many financial problems (Hadar and Seo 1988, Dionne and Li 2014). Li (2011) further extends the concept to higher-degree expectation dependence. Parallel to what we have done for stochastic dominance, we extend the concept of expectation dependence to a continuum of rules.

Definition 10. A random variable $\tilde{x}$ is **negatively** $(N+c)$**th-degree expectation dependent** on $\tilde{y}$, if and only if $Cov[\tilde{x}, u(\tilde{y})] \leq 0$ for all $u \in \mathscr{U}_{N,c}$.

Let $D^{(0)}(y;\tilde{x}) = (E[\tilde{x}] - E[\tilde{x}|\tilde{y} \leq y])\, Pr[\tilde{y} \leq y]$ and $D^{(n)}(y;\tilde{x}) = \int_a^y D^{(n-1)}(s;\tilde{x})ds$ for $n \geq 1$. Noticing that $Cov[\tilde{x}, u(\tilde{y})] = \int_a^b u^{(1)}(y) D^{(0)}(y;\tilde{x})dy$, we can follow the proof of Theorem 2 directly to obtain that $\tilde{x}$ is negatively $(N+c)$th-degree expectation dependent on $\tilde{y}$, if and only if

$$D^{(n)}(b;\tilde{x}) \geq 0, \quad n = 1, \cdots, N-1, \quad \int_a^y D^{(N-1)}(t;\tilde{x}) dh_c(t) \geq 0, \ \forall y.$$

The concept shown in Definition 10 can be applied to the risky choices under correlated background risk as analyzed by Tsetlin and Winkler (2005) and Li (2011). To be specific, let w_0 represent the initial wealth of an investor, and $\tilde{x}$ and $\tilde{y}$ denote the random net payoffs of a risky project and a non-tradable background risk which are correlated. Let α be the amount of money invested in the project. The optimal decision corresponds to

$$\alpha^* = \max_{\alpha} Eu(w_0 + \alpha\tilde{x} + \tilde{y}).$$

Given $u \in \mathscr{U}_{N+1,c}$, the problem is a concave problem and the solution is uniquely determined by the first-order condition $E[\tilde{x}u^{(1)}(w_0 + \alpha\tilde{x} + \tilde{y})] = 0$. Evaluating this condition at $\alpha = 0$ yields

$$E[\tilde{x}u^{(1)}(w_0 + \tilde{y})] = Cov\left[\tilde{x}, u^{(1)}(w_0 + \tilde{y})\right] + E[\tilde{x}]Eu^{(1)}(w_0 + \tilde{y}).$$

Since $E[\tilde{x}] > 0$ and $Eu^{(1)}(w_0 + \tilde{y}) > 0$, if $Cov\left[\tilde{x}, u^{(1)}(w_0 + \tilde{y})\right] \geq 0$, then the investor has a positive demand for the risky asset. Applying the concept of negatively $(N+c)$th-degree expectation dependence with $v(y) = -u^{(1)}(w_0 + y) \in \mathscr{U}_{N,c}$, we immediately obtain the following result.

Corollary 6. *Let $N \in \mathbb{N}$, $c \in (0,1)$, and $E[\tilde{x}] > 0$. If $\tilde{x}$ is negatively $(N+c)$th-degree expectation dependent on $\tilde{y}$, then $\alpha^* > 0$ for all $u \in \mathscr{U}_{N+1,c}$.*

[14] Alternatively, one can assume instead that $u_0^{(1)} > 0$, $u_0^{(2)} < 0$, $p^{(1)}(e) > 0$, $p^{(2)}(e) < 0$ and $u_1 \in \mathscr{U}_{N+1,c}$. Then, $EU(e, \tilde{\varepsilon}_i)$ is concave in e and the optimal level of precautionary effort is uniquely determined by the first-order condition $EU_e(e_i^*, \tilde{\varepsilon}_i) = 0$. Therefore, $e_2^* \geq e_1^*$ if and only if $EU_e(e_1^*, \tilde{\varepsilon}_2) \geq 0 = EU_e(e_1^*, \tilde{\varepsilon}_1)$, which after some algebraic manipulations turns out to be $Eu_1(w_1 + \tilde{\varepsilon}_2) + Eu_1(w_1 - l + \tilde{\varepsilon}_1) \geq Eu_1(w_1 + \tilde{\varepsilon}_1) + Eu_1(w_1 - l + \tilde{\varepsilon}_2)$. Since by Theorem 3, $[\tilde{\varepsilon}_2; -l + \tilde{\varepsilon}_1]$ (N+1+c)SD $[-l + \tilde{\varepsilon}_2; \tilde{\varepsilon}_1]$ if $\tilde{\varepsilon}_1$(N+c)SD$\tilde{\varepsilon}_2$, we come to the conclusion that for $u_1 \in \mathscr{U}_{N+1,c}$, $e_2^* \geq e_1^*$ if $\tilde{\varepsilon}_1$ (N+c)SD $\tilde{\varepsilon}_2$.

6.4. Ranking Income Distributions

Parallel to stochastic dominance which compares distributions according to their relative riskiness, a huge line of the literature concerning social welfare ranks income distributions according to their relative inequality. Since the seminal papers of Atkinson (1970) and Rothschild and Stiglitz (1970), these two lines of the literature have been in a tangle and could share new directions of research with each other.[15] In this section, we show how to apply our findings to compare income distributions.

Let F and G be cdfs of income $\tilde{y}$. Denote $F^{-1}(s) = \inf\{y : F(y) \geq s\}$ and $G^{-1}(s) = \inf\{y : G(y) \geq s\}$ as the inverse functions of F and G. Define the Lorenz curves of F and G as $L_F(p) = \frac{1}{E_F(\tilde{y})}\int_0^p F^{-1}(s)ds$ and $L_G(p) = \frac{1}{E_G(\tilde{y})}\int_0^p G^{-1}(s)ds$ (Gastwirth 1971). Let u denote the utility of the social planner and $Eu(\tilde{y})$ be the social welfare function. Atkinson (1970) has shown that $E_F u(\tilde{y}) \geq E_G u(\tilde{y})$ for all $u \in \mathscr{U}'_{2,0}$ if and only if $L_F(p) \geq L_G(p)$ for all $p \in [0,1]$. He indicated that this condition is equivalent to the second-degree increase in risk as shown in Section 3.3. Shorrocks (1983) further demonstrated that $E_F u(\tilde{y}) \geq E_G u(\tilde{y})$ for all $u \in \mathscr{U}_{2,0}$ if and only if $E_F(\tilde{y})L_F(p) \geq E_G(\tilde{y})L_G(p)$ for all $p \in [0,1]$. He defined this condition as generalized Lorenz dominance and showed that it is equivalent to SSD.

In the above discussions, concavity in utility is required. When a glass ceiling is a concern, a mean-preserving spread for the high end of the income distribution would be considered to be an improvement in social welfare or group inequality. In other words, it is possible to have the case where the social planner's preference is among $\mathscr{U}_{1,c}$. Our findings can help to rank income distributions according to (1+c)SD based on a Lorenz-type characterization.[16]

Corollary 7. *Let $c \in (0,1)$. F dominates G by (1+c)SD, if and only if $\int_0^p h_c(F^{-1}(s))ds \geq \int_0^p h_c(G^{-1}(s))ds$ for all $p \in [0,1]$.*

By adopting the Canadian Family Expenditure Survey, Barrett and Donald (2010) found that the income distribution in 1986 does not dominate that in 1978 at FSD but 1986 dominates 1978 at SSD. Corollary 7 provides a theoretical foundation for researchers to develop tests for (1+c)SD and could help to further examine whether 1986 dominates 1978 by (1+c)SD.

7. Conclusion

This paper develops a continuum of stochastic dominance rules, which encompass all the Nth-degree stochastic dominance rules with $N \in \mathbb{N}$. We have justified that the Nth-degree absolute

[15] See Gajdos and Weymark (2012) for a survey of inequality and risk.

[16] Muliere and Scarsini (1989, p. 317) note that the equivalence between stochastic dominance and inequality measures based on Lorenz curves does not hold any more for higher orders. Thus, how to extend our approach to compare income distributions for higher orders would be an interesting research question for future studies.

risk aversion can serve as a noble work horse to classify the risk attitude of individuals. The $(N+c)$th-degree stochastic dominance rules are defined as the consensus in the distribution ranking of individuals whose $(N+1)$th-degree absolute risk aversion shares the same lower bound, $(1-1/c)$. This definition yields tractable integral conditions, which allow us to generalize the existing results on stochastic dominance. We have extended the relationship between NSD and risk apportionment to (N+c)SD. Furthermore, we have illustrated how to use (1+c)SD to characterize the choice behavior of individuals with a limited extent of risk lovingness. In applications, we show that (N+c)SD can be readily used to analyze optimization problems involving precautionary saving, self-protection, risky investment, income inequality and the like.

Compared to prior studies, the major advance we make is to offer a new interpolation of integer-degree stochastic dominance rules with an economically sensible approach to classifying individuals. As stochastic dominance rules are useful in many economic contexts beyond what we have illustrated in this paper (Levy 1992), we expect that (N+c)SD can facilitate the comparative static analysis in future economic studies. Mathematically, the characterization of (N+c)SD simply introduces a weighting function to the integrand, which means that the existing statistical approaches for testing SSD such as Anderson (1996) or Davidson and Duclos (2000) can be extended to test (N+c)SD. Our stochastic dominance rule can also be employed in financial contexts. For example, we can develop a linear programming test for (N+c)SD in the same spirit as Post (2003) or Li and Linton (2010) to test for the stochastic dominance efficiency of a given portfolio; we can also use (N+c)SD to analyze aggregate investor preferences in the same way as Post and Levy (2005). A final remark is that we build up our theory under the expected utility framework. An extension to the rank-dependent utility framework would be fruitful for future studies.

Appendix

Proof of Theorem 1. We first prove the result with $N=2$, where $i \trianglerighteq_2 j$ amounts to saying that for any gamble $\tilde{x}$,

$$\exists w_i \ Eu_i(w_i+\tilde{x}) > u_i(w_i) \Longrightarrow \forall w_j \ Eu_j(w_j+\tilde{x}) > u_j(w_j).$$

Under $u_i^{(1)}, u_j^{(1)} > 0$, we will prove that $i \trianglerighteq_2 j$ if and only if $\min_w \rho_{u_i,2}(w) \geq \max_w \rho_{u_j,2}(w)$.

To prove the "if" part, suppose that $\min_w \rho_{u_i,2}(w) \geq \max_w \rho_{u_j,2}(w)$ and $Eu_i(w_i+\tilde{x}) > u_i(w_i)$ hold true. Define $\hat{u}_i(x) = [u_i(w_i+x) - u_i(w_i)]/u_i^{(1)}(w_i)$ and $\hat{u}_j(x) = [u_j(w_j+x) - u_j(w_j)]/u_j^{(1)}(w_j)$. Then, $\hat{u}_i$ and $\hat{u}_j$ satisfy $\hat{u}_i(0) = \hat{u}_j(0) = 0$, $\hat{u}_i^{(1)}(0) = \hat{u}_j^{(1)}(0) = 1$ and $\hat{u}_i^{(1)}, \hat{u}_j^{(1)} > 0$. In addition, $Eu_i(w_i+\tilde{x}) > u_i(w_i)$ is equivalent to $E\hat{u}_i(\tilde{x}) > 0$, and $Eu_j(w_j+\tilde{x}) > u_j(w_j)$, the fact we need to prove, is equivalent to $E\hat{u}_j(\tilde{x}) > 0$. Noticing that $\rho_{\hat{u}_i,2}(x) = \rho_{u_i,2}(w_i+x)$ and $\rho_{\hat{u}_j,2}(x) = \rho_{u_j,2}(w_i+x)$, we have $\rho_{\hat{u}_i,2}(x) \geq \rho_{\hat{u}_j,2}(x)$ for all $x \in \mathbb{R}$ under $\min_w \rho_{u_i,2}(w) \geq \max_w \rho_{u_j,2}(w)$. Thus, we have for $x \geq 0$ that

$$\hat{u}_j(x) = \int_0^x e^{\log \hat{u}_j^{(1)}(y)} dy = \int_0^x e^{-\int_0^y \rho_{\hat{u}_j,2}(z)dz} dy \geq \int_0^x e^{-\int_0^y \rho_{\hat{u}_i,2}(z)dz} dy = \hat{u}_i(x),$$

and for $x < 0$ that

$$\hat{u}_j(x) = -\int_{-|x|}^0 e^{\log \hat{u}_j^{(1)}(y)} dy = -\int_{-|x|}^0 e^{\int_y^0 \rho_{\hat{u}_j,2}(z)dz} dy \geq -\int_{-|x|}^0 e^{\int_y^0 \rho_{\hat{u}_i,2}(z)dz} dy = \hat{u}_i(x).$$

That is, we always have $\hat{u}_j(x) \geq \hat{u}_i(x)$, yielding $E\hat{u}_j(\tilde{x}) \geq E\hat{u}_i(\tilde{x}) > 0$.

To prove the "only if" part, we argue by contradiction. Suppose there exist w_i, w_j such that $\rho_{u_i,2}(w_i) < \rho_{u_j,2}(w_j)$. By defining $\hat{u}_i(x)$ and $\hat{u}_j(x)$ as shown above, we then obtain $\rho_{\hat{u}_i,2}(0) < \rho_{\hat{u}_j,2}(0)$, which implies by continuity the existence of $\delta > 0$ such that $\rho_{\hat{u}_i,2}(x) < \rho_{\hat{u}_j,2}(x)$ if $|x| < \delta$. This in turn implies that $\hat{u}_i(x) > \hat{u}_j(x)$ for $0 \neq |x| < \delta$. Noticing that $\hat{u}_i\left(\frac{\delta}{2}\right) > 0 > \hat{u}_i\left(-\frac{\delta}{2}\right)$, we introduce a Bernoulli lottery $\tilde{x}_\varepsilon$ that takes $\varepsilon + \frac{\delta}{2}$ with probability $|\hat{u}_i\left(-\frac{\delta}{2}\right)| / \left(\hat{u}_i\left(\frac{\delta}{2}\right) + |\hat{u}_i\left(-\frac{\delta}{2}\right)|\right)$ and takes $-\frac{\delta}{2}$ with probability $\hat{u}_i\left(\frac{\delta}{2}\right) / \left(\hat{u}_i\left(\frac{\delta}{2}\right) + |\hat{u}_i\left(-\frac{\delta}{2}\right)|\right)$. It is obvious that for $\varepsilon = 0$, $E\hat{u}_i(\tilde{x}_0) = 0 > E\hat{u}_j(\tilde{x}_0)$. By continuity and strict monotonicity, we increase ε a little to obtain $E\hat{u}_i(\tilde{x}_\varepsilon) > 0 > E\hat{u}_j(\tilde{x}_\varepsilon)$, leading us to $Eu_i(w_i+\tilde{x}_\varepsilon) > u_i(w_i)$ but $Eu_j(w_j+\tilde{x}_\varepsilon) < u_j(w_j)$, a contradiction to the definition of $i \trianglerighteq_2 j$.

For $N \geq 3$, we introduce $v(w) = Eu\left(w + A_{N-2}(\tilde{\varepsilon})\right) - Eu\left(w + B_{N-2}(\tilde{\varepsilon})\right)$. Simple algebraic manipulations demonstrate that $i \trianglerighteq_N j$ amounts to saying that for any gamble $\tilde{x}$,

$$\exists w_i, \ Ev_i(w_i+\tilde{x}) > Ev_i(w_i) \Longrightarrow \forall w_j, \ Ev_j(w_j+\tilde{x}) > Ev_j(w_j).$$

By applying the former result with $N=2$ to v_i and v_j, we immediately obtain that under $v_i^{(1)}, v_j^{(1)} > 0$, $i \trianglerighteq_N j$ if and only if $\min_w \rho_{v_i,2}(w) \geq \max_w \rho_{v_j,2}(w)$. The proof will be completed if we can prove $(-1)^N u_i^{(N-1)}, (-1)^N u_j^{(N-1)} > 0$ implies that $v_i^{(1)}, v_j^{(1)} > 0$; and moreover,

$$\forall k > 0, \ \forall \tilde{\varepsilon}, \ \min_w \rho_{v_i,2}(w) \geq \max_w \rho_{v_j,2}(w) \Longleftrightarrow \min_w \rho_{u_i,N}(w) \geq \max_w \rho_{u_j,N}(w). \tag{A1}$$

Notice that $(-1)^N u^{(N-1)} > 0$ is equivalent to $(-1)^{N-1}\left[-u^{(1)}\right]^{(N-2)} > 0$. According to Eeckhoudt and Schlesinger (2006, p. 286), this further implies that

$$-Eu^{(1)}u\left(w+B_{N-2}\left(\tilde{\varepsilon}\right)\right) \geq -Eu^{(1)}u\left(w+A_{N-2}\left(\tilde{\varepsilon}\right)\right),$$

which confirms $v^{(1)} > 0$. Thus, the remaining part is devoted to proving (A1).

For $N=3$, $v(w)=u(w-k)-u(w)$ and $\rho_{v,2}(w) = -\frac{u^{(2)}(w-k)-u^{(2)}(w)}{u^{(1)}(w-k)-u^{(1)}(w)}$. Moreover, $u^{(2)} < 0$ implies that $v^{(1)} > 0$. We show that the following two statements are equivalent:

(i) $\rho_{v_i,2}(w_i) \geq \rho_{v_j,2}(w_j)$ holds true for any $k>0$ and any $w_i, w_j \in \mathbb{R}$;

(ii) $\rho_{u_i,3}(w_i) \geq \rho_{u_j,3}(w_j)$ for any $w_i, w_j \in \mathbb{R}$.

Letting $k \to 0$ yields $\rho_{v,2}(w) \to -\frac{u^{(3)}(w)}{u^{(2)}(w)} = \rho_{u,3}(w)$, which confirms that (i) implies (ii). On the contrary, by applying Cauchy's mean value theorem to $\rho_{v,2}(w)$, we have $\rho_{v,2}(w) = -\frac{u^{(3)}(w-\theta k)}{u^{(2)}(w-\theta k)}$ with $\theta \in (0,1)$, which confirms that (ii) implies (i). This proves (A1) for $N=3$.

For $N=4$, $v(w)=Eu\left(w+\tilde{\varepsilon}\right)-u(w)$ and $\rho_{v,2}(w) = -\frac{Eu^{(2)}(w+\tilde{\varepsilon})-u^{(2)}(w)}{Eu^{(1)}(w+\tilde{\varepsilon})-u^{(1)}(w)}$. Moreover, $u^{(3)} > 0$ implies that $v^{(1)} > 0$. We show that the following two statements are equivalent:

(i) $\rho_{v_i,2}(w_i) \geq \rho_{v_j,2}(w_j)$ holds true for any pure risk $\tilde{\varepsilon}$ and any $w_i, w_j \in \mathbb{R}$;

(ii) $\rho_{u_i,4}(w_i) \geq \rho_{u_j,4}(w_j)$ for any $w_i, w_j \in \mathbb{R}$.

Substituting $\tilde{\varepsilon} = [-k;k]$ and letting $k \to 0$ yields $\rho_{v,2}(w) \to -\frac{u^{(4)}(w)}{u^{(3)}(w)} = \rho_{u,4}(w)$, which confirms that (i) implies (ii). On the contrary, by applying Cauchy's mean value theorem twice to $\rho_{v,2}(w)$, we have $\rho_{v,2}(w) = -\frac{E\left[u^{(3)}(w+\theta_1\tilde{\varepsilon})\tilde{\varepsilon}\right]}{E\left[u^{(2)}(w+\theta_1\tilde{\varepsilon})\tilde{\varepsilon}\right]} = -\frac{E\left[u^{(4)}(w+\theta_1\theta_2\tilde{\varepsilon})\tilde{\varepsilon}^2\right]}{E\left[u^{(3)}(w+\theta_1\theta_2\tilde{\varepsilon})\tilde{\varepsilon}^2\right]}$ with $\theta_1, \theta_2 \in (0,1)$. Based on it, it becomes apparent that (ii) implies (i). This proves (A1) for $N=4$.

The proof for $N \geq 5$ is based on recursion and induction. Indeed, by inserting $A_{N-2}(\tilde{\varepsilon}) = [B_{N-4}(\tilde{\varepsilon}_1); \tilde{\varepsilon}_2 + A_{N-4}(\tilde{\varepsilon}_1)]$ and $B_{N-2}(\tilde{\varepsilon}) = [A_{N-2}(\tilde{\varepsilon}_1); \tilde{\varepsilon}_2 + B_{N-4}(\tilde{\varepsilon}_1)]$ into $v(w) = Eu\left(w+A_{N-2}\left(\tilde{\varepsilon}\right)\right) - Eu\left(w+B_{N-2}\left(\tilde{\varepsilon}\right)\right)$, we obtain $v(w) = E\bar{v}(w+\tilde{\varepsilon}_2) - \bar{v}(w)$, where $\bar{v}(w) = Eu(w+A_{N-4}(\tilde{\varepsilon}_1)) - Eu(w+B_{N-4}(\tilde{\varepsilon}_2))$. We show that the following two statements are equivalent:

(i) $\rho_{v_i,2}(w_i) \geq \rho_{v_j,2}(w_j)$ holds true for any pure risks $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$ and any $w_i, w_j \in \mathbb{R}$;

(ii) $\rho_{u_i,N}(w_i) \geq \rho_{u_j,N}(w_j)$ for any $w_i, w_j \in \mathbb{R}$.

Notice that $(-1)^N u^{(N-1)} > 0$ is equivalent to $(-1)^{N-3}\left[-u^{(3)}\right]^{(N-4)} > 0$. According to Eeckhoudt and Schlesinger (2006, p. 286), this further implies that

$$-Eu^{(3)}u\left(w+B_{N-4}\left(\tilde{\varepsilon}_1\right)\right) \geq -Eu^{(3)}u\left(w+A_{N-4}\left(\tilde{\varepsilon}_1\right)\right),$$

which confirms $\bar{v}^{(3)} > 0$. By applying the result for $N=4$ here, we see that (i) is equivalent to

(iii) $\rho_{\bar{v}_i,4}(w_i) \geq \rho_{\bar{v}_j,4}(w_j)$ for any pure risk $\tilde{\varepsilon}_1$ and any $w_i, w_j \in \mathbb{R}$.

Since $\rho_{\bar{v},4}(w) = \rho_{\bar{v}^{(2)},2}(w)$, the above is equivalent to

(iv) $\rho_{\bar{v}_i^{(2)},2}(w_i) \geq \rho_{\bar{v}_j^{(2)},2}(w_j)$ for any pure risk $\tilde{\varepsilon}_1$ and any $w_i, w_j \in \mathbb{R}$.

Notice that $(-1)^N u^{(N-1)} > 0$ is equivalent to $(-1)^{N-2}\left[u^{(2)}\right]^{(N-3)} > 0$. By applying the result for $N-2$ here, we see that (iv) holds true if and only if

(v) $\rho_{u_i^{(2)},N-2}(w_i) \geq \rho_{u_j^{(2)},N-2}(w_j)$ for any $w_i, w_j \in \mathbb{R}$.

Due to $\rho_{u^{(2)},N-2}(w) = \rho_{u,N}(w)$, (v) is equivalent to (ii). This proves (A1) for $N \geq 5$. □

Proof of Theorem 2. Integrating $Eu(\tilde{x}) - Eu(\tilde{y}) = \int_a^b u^{(1)}(x)[G(x) - F(x)]dx$ by parts yields

$$\begin{aligned} &Eu(\tilde{x}) - Eu(\tilde{y}) \\ =&\sum_{n=1}^{N-1}(-1)^{n+1}u^{(n)}(b)[G^{(n)}(b) - F^{(n)}(b)] + \int_a^b (-1)^{N+1}u^{(N)}(x)\left[G^{(N-1)}(x) - F^{(N-1)}(x)\right]dx. \end{aligned} \tag{A2}$$

Letting $\Lambda(x) = \int_a^x [G^{(N-1)}(t) - F^{(N-1)}(t)]dh_c(t)$ and integrating the last term by parts yields

$$\begin{aligned} &\int_a^b (-1)^{N+1}u^{(N)}(x)\left[G^{(N-1)}(x) - F^{(N-1)}(x)\right]dx \\ =&(-1)^{N+1}u^{(N)}(b)\frac{\Lambda(b)}{h_c^{(1)}(b)} + \int_a^b \left[(-1)^{N+2}u^{(N+1)} + \left(\frac{1}{c} - 1\right)(-1)^{N+1}u^{(N)}\right](x)\frac{\Lambda(x)}{h_c^{(1)}(x)}dx. \end{aligned}$$

Inserting it into the above, we obtain

$$\begin{aligned} Eu(\tilde{x}) - Eu(\tilde{y}) =& \sum_{n=1}^{N-1}(-1)^{n+1}u^{(n)}(b)[G^{(n)}(b) - F^{(n)}(b)] + (-1)^{N+1}u^{(N)}(b)\frac{\Lambda(b)}{h_c^{(1)}(b)} \\ &+ \int_a^b \left[(-1)^{N+2}u^{(N+1)} + \left(\frac{1}{c} - 1\right)(-1)^{N+1}u^{(N)}\right](x)\frac{\Lambda(x)}{h_c^{(1)}(x)}dx, \end{aligned}$$

from which it becomes obvious that $Eu(\tilde{x}) \geq Eu(\tilde{y})$ for all $u \in \mathscr{U}_{N,c}$ if and only if $G^{(n)}(b) \geq F^{(n)}(b)$ for $n = 1, \cdots, N-1$ and $\Lambda(x) \geq 0$ for all x. □

Proof of Theorem 3. By definition, we just need to verify

$$Eu(\tilde{x}_{N+c} + \tilde{y}_M) + Eu(\tilde{y}_{N+c} + \tilde{x}_M) \geq Eu(\tilde{x}_{N+c} + \tilde{x}_M) + Eu(\tilde{y}_{N+c} + \tilde{y}_M) \tag{A3}$$

for all $u \in \mathscr{U}_{N+M,c}$. By letting $v(w) = Eu(w + \tilde{y}_M) - Eu(w + \tilde{x}_M)$, (A5) translates into $Ev(\tilde{x}_{N+c}) \geq Ev(\tilde{y}_{N+c})$. If we can show that $v \in \mathscr{U}_{N,c}$, then the desired result follows straightforwardly. Denote the cdfs of $\tilde{x}_M$ and $\tilde{y}_M$ by F and G, respectively. Integrating $(-1)^{n+1}v^{(n)}(w) = (-1)^{n+1}\left[Eu^{(n)}(w + \tilde{y}_M) - Eu^{(n)}(w + \tilde{x}_M)\right]$ by parts, we obtain

$$\begin{aligned} (-1)^{n+1}v^{(n)}(w) =& \sum_{j=1}^{M-1}(-1)^{n+j+1}u^{(n+j)}(w+b)\left[G^{(j)}(b) - F^{(j)}(b)\right] \\ &+ \int_a^b (-1)^{n+M+1}u^{(n+M)}(w+x)\left[G^{(M-1)}(x) - F^{(M-1)}(x)\right]dx. \end{aligned}$$

For $n = 1, \cdots, N$, thanks to $\tilde{x}_M$ MSD $\tilde{y}_M$ and $(-1)^{n+1}u^{(n)} \geq 0$ with $n = 1, \cdots, N+M$, every term in the above is positive and hence $(-1)^{n+1}v^{(n)}(w) \geq 0$ for all $n = 1, \cdots, N$. In particular,

$$\begin{aligned} (-1)^{N+1}v^{(N)}(w) &\geq \int_a^b (-1)^{N+M+1}u^{(N+M)}(w+x)\left[G^{(M-1)}(x) - F^{(M-1)}(x)\right]dx, \\ (-1)^{N+2}v^{(N+1)}(w) &\geq \int_a^b (-1)^{N+M+2}u^{(N+M+1)}(w+x)\left[G^{(M-1)}(x) - F^{(M-1)}(x)\right]dx. \end{aligned}$$

Inserting $(-1)^{N+M+2}u^{(N+M+1)}+(1/c-1)(-1)^{N+M+1}u^{(N+M)}\geq 0$ into the above, we obtain that $(-1)^{N+2}v^{(N+1)}+(1/c-1)(-1)^{N+1}v^{(N)}\geq 0$, which verifies that $v\in\mathscr{U}_{N,c}$. □

Proof of Theorem 4. By definition, we just need to verify

$$Eu(\tilde{x}_{N+c_1}+\tilde{y}_{M+c_2})+Eu(\tilde{y}_{N+c_1}+\tilde{x}_{M+c_2})\geq Eu(\tilde{x}_{N+c_1}+\tilde{x}_{M+c_2})+Eu(\tilde{y}_{N+c_1}+\tilde{y}_{M+c_2}) \tag{A4}$$

for all $u\in\mathscr{U}_{N+M+1,c}$. By letting $v(w)=Eu(w+\tilde{y}_{M+c_2})-Eu(w+\tilde{x}_{M+c_2})$, (A4) translates into $Ev(\tilde{x}_{N+c_1})\geq Ev(\tilde{y}_{N+c_1})$. If we can show that $v\in\mathscr{U}_{N,c_1}$, then the desired result follows straightforwardly. Denote the cdfs of $\tilde{x}_{M+c_2}$ and $\tilde{y}_{M+c_2}$ by F and G, respectively. Letting $\Lambda(x)=\int_a^x[G^{(M-1)}(t)-F^{(M-1)}(t)]dh_{c_2}(t)$ and integrating $(-1)^{n+1}v^{(n)}(w)=(-1)^{n+1}\left[Eu^{(n)}(w+\tilde{y}_{M+c_2})-Eu^{(n)}(w+\tilde{x}_{M+c_2})\right]$ by parts, we obtain

$$\begin{aligned}&(-1)^{n+1}v^{(n)}(w)\\&=\sum_{j=1}^{M-1}(-1)^{n+j+1}u^{(n+j)}(w+b)[G^j(b)-F^j(b)]+(-1)^{n+M+1}u^{(n+M)}(w+b)\frac{\Lambda(b)}{h_{c_2}^{(1)}(b)}\\&\quad+\int_a^b\left[(-1)^{n+M+2}u^{(n+M+1)}+\left(\frac{1}{c_2}-1\right)(-1)^{n+M+1}u^{(n+M)}\right](w+x)\frac{\Lambda(x)}{h_{c_2}^{(1)}(x)}dx.\end{aligned}$$

For $n=1,\cdots,N$, thanks to $\tilde{x}_{M+c_2}$ (M+c2)SD $\tilde{y}_{M+c_2}$ and $(-1)^{n+1}u^{(n)}\geq 0$ with $n=1,\cdots,N+M+1$, every term in the above is positive and hence $(-1)^{n+1}v^{(n)}(w)\geq 0$ for all $n=1,\cdots,N$. In particular,

$$\begin{aligned}(-1)^{N+1}v^{(N)}(w)&\geq\int_a^b\left[(-1)^{N+M+2}u^{(N+M+1)}+\left(\frac{1}{c_2}-1\right)(-1)^{N+M+1}u^{(N+M)}\right](w+x)\frac{\Lambda(x)}{h_{c_2}^{(1)}(x)}dx\\&\geq\int_a^b(-1)^{N+M+2}u^{(N+M+1)}(w+x)\frac{\Lambda(x)}{h_{c_2}^{(1)}(x)}dx,\\(-1)^{N+2}v^{(N+1)}(w)&\geq\int_a^b\left[(-1)^{N+M+3}u^{(N+M+2)}+\left(\frac{1}{c_2}-1\right)(-1)^{N+M+2}u^{(N+M+1)}\right](w+x)\frac{\Lambda(x)}{h_{c_2}^{(1)}(x)}dx.\end{aligned}$$

Inserting $(-1)^{N+M+3}u^{(N+M+2)}+(1/c-1)(-1)^{N+M+2}u^{(N+M+1)}\geq 0$ into the above and using

$$\frac{1}{c}-1=\left(\frac{1}{c_1}-1\right)+\left(\frac{1}{c_2}-1\right),$$

we obtain $(-1)^{N+2}v^{(N+1)}+(1/c_1-1)(-1)^{N+1}v^{(N)}\geq 0$, which verifies that $v\in\mathscr{U}_{N,c_1}$. □

Proof of Corollary 1. By definition, we need to verify that

$$Eu(\tilde{x}_{N+c}+\tilde{y}_M)+Eu(\tilde{y}_{N+c}+\tilde{x}_M)\geq Eu(\tilde{x}_{N+c}+\tilde{x}_M)+Eu(\tilde{y}_{N+c}+\tilde{y}_M) \tag{A5}$$

for all $u\in\mathscr{U}'_{N+M,c}$. By letting $v(w)=Eu(w+\tilde{y}_M)-Eu(w+\tilde{x}_M)$, (A5) translates into $Ev(\tilde{x}_{N+c})\geq Ev(\tilde{y}_{N+c})$. If we can show that $v\in\mathscr{U}'_{N,c}$, then the desired result follows straightforwardly. Denote the cdfs of $\tilde{x}_M$ and $\tilde{y}_M$ by F and G, respectively. Since $\tilde{y}_M$ MIR $\tilde{x}_M$, we can follow the proof of Theorem 3 to obtain

$$(-1)^{n+1}v^{(n)}(w)=\int_a^b(-1)^{n+M+1}u^{(n+M)}(w+x)\left[G^{(M-1)}(x)-F^{(M-1)}(x)\right]dx.$$

In particular,

$$(-1)^{N+1}v^{(N)}(w) = \int_a^b (-1)^{N+M+1}u^{(N+M)}(w+x)\left[G^{(M-1)}(x) - F^{(M-1)}(x)\right]dx,$$
$$(-1)^{N+2}v^{(N+1)}(w) = \int_a^b (-1)^{N+M+2}u^{(N+M+1)}(w+x)\left[G^{(M-1)}(x) - F^{(M-1)}(x)\right]dx.$$

Inserting $(1-c)(-1)^{N+M+1}u^{(N+M)} + c(-1)^{N+M+2}u^{(N+M+1)} \geq 0$ into the above, we obtain $(1-c)(-1)^{N+1}v^{(N)} + c(-1)^{N+2}v^{(N+1)} \geq 0$, which verifies that $v \in \mathscr{U}'_{N,c}$. □

Proof of Theorem 5. Following the idea of Müller et al. (2016, Section 2.2), we only need to verify that when gambles $\tilde{x}$ and $\tilde{y}$ assume a finite number of values, $\tilde{x}$ (1+c)SD $\tilde{y}$ if and only if the distribution of $\tilde{x}$, F, can be obtained from the distribution of $\tilde{y}$, G, via a finite sequence of c-transfers and increasing transfers. The "if" part is obvious. To show the "only if" part, we follow the proof of Theorem 2.7 in Müller et al. (2016, p. 12) and define

$$A^+(p) = \int_0^p (h_c(F^{-1}(s)) - h_c(G^{-1}(s)))_+ ds,\ A^-(p) = \int_0^p (h_c(F^{-1}(s)) - h_c(G^{-1}(s)))_- ds,$$

where $x_+ = \max\{x, 0\}$ and $x_- = -\min\{x, 0\}$. For $a \in (0, A^-(1))$, let $\alpha(a)$ and $\beta(a)$ be the smallest probabilities that solve $A^+(\alpha(a)) = a$ and $A^-(\beta(a)) = a$. It follows from our Corollary 7 that $A^-(p) \leq A^+(p)$ for all $p \in [0,1]$ and hence $\alpha(a) \leq \beta(a)$ for all $a \in (0, A^-(1))$. We set

$$x_1(a) = G^{-1}(\alpha(a)),\ x_2(a) = F^{-1}(\alpha(a)),\ x_3(a) = F^{-1}(\beta(a)),\ x_4(a) = G^{-1}(\beta(a)).$$

Since $\tilde{x}$ and $\tilde{y}$ assume a finite number of values, then there is a finite sequence $0 = a_1 < a_2 < \cdots < a_k \leq A^-(1)$ such that the functions $a \to x_1(a), \cdots, x_4(a)$ are constant on (a_{i-1}, a_i). Denote the corresponding values of these functions as $x_{l,i} = x_l(a)$ for $a \in (a_{i-1}, a_i)$ and $l = 1, \cdots, 4$. In addition, at the points $x_{1,i}$ and $x_{4,i}$ the function G has jumps of sizes of at least $\eta_{1,i}$ and $\eta_{2,i}$, and at the corresponding points $x_{2,i}$ and $x_{3,i}$, the function F has jumps of sizes of at least $\eta_{1,i}$ and $\eta_{2,i}$, where $\eta_{1,i}$ and $\eta_{2,i}$ are given by the equation $\eta_{1,i}[e^{(1/c-1)x_{2,i}} - e^{(1/c-1)x_{1,i}}] = \eta_{2,i}[e^{(1/c-1)x_{4,i}} - e^{(1/c-1)x_{3,i}}]$. For $x \geq x_{4,k}$, we have $G(x) \geq F(x)$. Thus, F is obtained from G by a sequence of c-transfers described by the corresponding x's and η's above, plus a finite number of increasing transfers moving the mass from G to F right of $x_{4,k}$.

When $\tilde{x}$ (1+c)SD $\tilde{y}$, let $\Lambda(x) = \int_a^x [G(t) - F(t)]\, dh_c(t)$ and write

$$E[\tilde{x}] - E[\tilde{y}] = \int_a^b [G(x) - F(x)]dx = \int_a^b \frac{d\Lambda(x)}{h_c(x)} = \frac{\Lambda(b)}{h_c(b)} + \left(\frac{1}{c} - 1\right)\int_a^b \frac{\Lambda(x)}{h_c(x)}dx \geq 0.$$

Given $\Lambda(x) \geq 0$ for all x, the above equals zero if and only if $F(x) = G(x)$ almost everywhere. □

Proof of Theorem 6. Recall that $\tilde{x}$ (1+c)SD $\tilde{y}$ if and only if $\int_a^x [G(t) - F(t)]dh_c(t) \geq 0$ for all x. Taking $x = b$, we see that a necessary condition is

$$\int_a^b [G(t) - F(t)]dh_c(t) = \int_a^b h_c(t)dF(t) - \int_a^b h_c(t)dG(t) = E[h_c(\tilde{x})] - E[h_c(\tilde{y})] \geq 0.$$

To show the sufficiency of this condition, we distinguish between two cases. For $x \le \hat{x}$, it is obvious that $\int_a^x [G(t)-F(t)]dh_c(t) \ge 0$. For $x > \hat{x}$, we write $\int_a^x [G(t)-F(t)]dh_c(t) = \int_a^b [G(t)-F(t)]dh_c(t) - \int_x^b [G(t)-F(t)]dh_c(t)$, where the first term is nonnegative and the second term is negative. The overall term is nonnegative, justifying the sufficiency. □

Proof of Theorem 7. Let $\Lambda(x) = \int_x^b [G(t) - F(t)]dh_c^*(t)$. Integrating $Eu(\tilde{x}) - Eu(\tilde{y}) = \int_a^b u^{(1)}(x)[G(x)-F(x)]dx$ by parts yields

$$Eu(\tilde{x}) - Eu(\tilde{y}) = u^{(1)}(a)\frac{\Lambda(a)}{h_c^{*(1)}(a)} + \int_a^b \left[u^{(2)} + \left(\frac{1}{c} - 1\right)u^{(1)}\right](x)\frac{\Lambda(x)}{h_c^{*(1)}(x)}dx,$$

from which it is easily seen that $Eu(\tilde{x}) \ge Eu(\tilde{y})$ for all $u \in \mathscr{U}_{1,c}^*$ if and only if $\Lambda(x) \ge 0$ for all x. □

Proof of Theorem 8. Note that $i \trianglerighteq_2^r j$ amounts to saying that for any gamble $\tilde{x} > 0$,

$$\exists w_i \;\; Eu_i(w_i\tilde{x}) > u_i(w_i) \Longrightarrow \forall w_j \;\; Eu_j(w_j\tilde{x}) > u_j(w_j).$$

To prove the "if" part, suppose that $\min_w \rho_{u_i,2}^r(w) \ge \max_w \rho_{u_j,2}^r(w)$ and $Eu_i(w_i\tilde{x}) > u_i(w_i)$ hold true. Define $\hat{u}_i(x) = [u_i(w_i x) - u_i(w_i)]/[w_i u_i^{(1)}(w_i)]$ and $\hat{u}_j(x) = [u_j(w_j x) - u_j(w_j)]/[w_j u_j^{(1)}(w_j)]$. Then, $\hat{u}_i$ and $\hat{u}_j$ satisfy $\hat{u}_i(1) = \hat{u}_j(1) = 0$, $\hat{u}_i^{(1)}(1) = \hat{u}_j^{(1)}(1) = 1$ and $\hat{u}_i^{(1)}, \hat{u}_j^{(1)} > 0$. In addition, $Eu_i(w_i\tilde{x}) > u_i(w_i)$ is equivalent to $E\hat{u}_i(\tilde{x}) > 0$, and $Eu_j(w_j\tilde{x}) > u_j(w_j)$, the fact we need to prove, is equivalent to $E\hat{u}_j(\tilde{x}) > 0$. Noticing that for $k = i, j$,

$$\rho_{\hat{u}_k,2}(x) = -\frac{\hat{u}_k^{(2)}(x)}{\hat{u}_k^{(1)}(x)} = -\frac{w_k u_k^{(2)}(w_k x)}{u_k^{(1)}(w_k x)} = \frac{1}{x}\rho_{u_k,2}^r(w_k x),$$

we have $\rho_{\hat{u}_i,2}(x) \ge \rho_{\hat{u}_j,2}(x)$ for all $x > 0$ under $\min_{w>0} \rho_{u_i,2}^r(w) \ge \max_{w>0} \rho_{u_j,2}^r(w)$. Thus, we have for $x \ge 1$ that

$$\hat{u}_j(x) = \int_1^x e^{\log \hat{u}_j^{(1)}(y)} dy = \int_1^x e^{-\int_1^y \rho_{\hat{u}_j,2}(z)dz} dy \ge \int_1^x e^{-\int_1^y \rho_{\hat{u}_j,2}(z)dz} dy = \hat{u}_i(x),$$

and for $x < 1$ that

$$\hat{u}_j(x) = -\int_x^1 e^{\log \hat{u}_j^{(1)}(y)} dy = -\int_x^1 e^{\int_y^1 \rho_{\hat{u}_j,2}(z)dz} dy \ge -\int_x^1 e^{\int_y^1 \rho_{\hat{u}_i,2}(z)dz} dy = \hat{u}_i(x).$$

That is, we always have $\hat{u}_j(x) \ge \hat{u}_i(x)$, yielding $E\hat{u}_j(\tilde{x}) \ge E\hat{u}_i(\tilde{x}) > 0$.

To prove the "only if" part, we argue by contradiction. Suppose there exist w_i, w_j such that $\rho_{u_i,2}^r(w_i) < \rho_{u_j,2}^r(w_j)$. By defining $\hat{u}_i(x)$ and $\hat{u}_j(x)$ as shown above, we then obtain $\rho_{\hat{u}_i,2}(1) < \rho_{\hat{u}_j,2}(1)$, which implies by continuity the existence of $\delta > 0$ such that $\rho_{\hat{u}_i,2}(x) < \rho_{\hat{u}_j,2}(x)$ if $|x-1| < \delta$. This in turn implies that $\hat{u}_i(x) > \hat{u}_j(x)$ for $0 \ne |x - 1| < \delta$. Noticing that $\hat{u}_i\left(1 + \frac{\delta}{2}\right) > 0 > \hat{u}_i\left(1 - \frac{\delta}{2}\right)$, we introduce a Bernoulli lottery $\tilde{x}_\varepsilon$ that takes $1 + \varepsilon + \frac{\delta}{2}$ with probability $|\hat{u}_i\left(1 - \frac{\delta}{2}\right)| / \left(\hat{u}_i\left(1 + \frac{\delta}{2}\right) + |\hat{u}_i\left(1 - \frac{\delta}{2}\right)|\right)$ and takes $1 - \frac{\delta}{2}$ with probability $\hat{u}_i\left(1 + \frac{\delta}{2}\right) / \left(\hat{u}_i\left(1 + \frac{\delta}{2}\right) + |\hat{u}_i\left(1 - \frac{\delta}{2}\right)|\right)$. For $\varepsilon = 0$, $E\hat{u}_i(\tilde{x}_0) = 0 > E\hat{u}_j(\tilde{x}_0)$. By continuity and strict monotonicity, we increase ε a little to obtain $E\hat{u}_i(\tilde{x}_\varepsilon) > 0 > E\hat{u}_j(\tilde{x}_\varepsilon)$, leading us to $Eu_i(w_i\tilde{x}_\varepsilon) > u_i(w_i)$ but $Eu_j(w_j\tilde{x}_\varepsilon) < u_j(w_j)$, a contradiction to $i \trianglerighteq_2^r j$. $\square$

Proof of Theorem 9. Let $\Lambda(x) = \int_a^x [G(t) - F(t)] dh_c^r(t)$. Integrating $Eu(\tilde{x}) - Eu(\tilde{y}) = \int_a^b u^{(1)}(x)[G(x) - F(x)]dx$ by parts yields

$$Eu(\tilde{x}) - Eu(\tilde{y}) = u^{(1)}(b)\frac{\Lambda(b)}{h_c^{r(1)}(b)} + \int_a^b \left[-u^{(2)}(x) + \left(\frac{1}{c} - 1\right)\frac{u^{(1)}(x)}{x}\right]\frac{\Lambda(x)}{h_c^{r(1)}(x)}dx,$$

from which it is easily seen that $Eu(\tilde{x}) \geq Eu(\tilde{y})$ for all $u \in \mathscr{U}_{1,c}^r$ if and only if $\Lambda(x) \geq 0$ for all x. $\square$

Proof of Corollary 7. We will show that (i) "$\int_a^x F(t)dh_c(t) \leq \int_a^x G(t)dh_c(t)$ for all x" is equivalent to (ii) "$\int_0^p h_c(F^{-1}(s))ds \geq \int_0^p h_c(G^{-1}(s))ds$ for all p". We first show that (i) implies (ii). Rewrite

$$\int_0^p h_c(F^{-1}(s))ds = \int_a^{F^{-1}(p)} h_c(t)dF(t) = h_c(F^{-1}(p))p - \int_a^{F^{-1}(p)} F(t)dh_c(t),$$
$$\int_0^p h_c(G^{-1}(s))ds = \int_a^{G^{-1}(p)} h_c(t)dG(t) = h_c(G^{-1}(p))p - \int_a^{G^{-1}(p)} G(t)dh_c(t).$$

If $F^{-1}(p) = G^{-1}(p)$, it is straightforward to see that (i) implies (ii). If $F^{-1}(p) > G^{-1}(p)$, we rewrite $\int_0^p h_c(F^{-1}(s))ds = \left(h_c(G^{-1}(p))p - \int_a^{G^{-1}(p)} F(t)dh_c(t)\right) + [h_c(F^{-1}(p)) - h_c(G^{-1}(p))]p - \int_{G^{-1}(p)}^{F^{-1}(p)} F(t)dh_c(t)$, where the term in the first set of brackets is not smaller than $\int_0^p h_c(G^{-1}(s))ds$, and the residual term is nonnegative due to $F(t) \leq p$ for $t \in [G^{-1}(p), F^{-1}(p)]$. If $F^{-1}(p) < G^{-1}(p)$, we rewrite $\int_0^p h_c(F^{-1}(s))ds = \left(h_c(G^{-1}(p))p - \int_a^{G^{-1}(p)} F(t)dh_c(t)\right) + \int_{F^{-1}(p)}^{G^{-1}(p)} F(t)dh_c(t) - [h_c(G^{-1}(p)) - h_c(F^{-1}(p))]p$, where the term in the first set of brackets is not smaller than $\int_0^p h_c(G^{-1}(s))ds$, and the residual term is nonnegative due to $F(t) \geq p$ for $t \in [F^{-1}(p), G^{-1}(p)]$. Overall, we obtain that (i) implies (ii). Next, we show that (ii) implies (i). Rewrite

$$\int_a^x F(t)dh_c(t) = F(x)h_c(x) - \int_a^x h_c(t)dF(t) = F(x)h_c(x) - \int_0^{F(x)} h_c(F^{-1}(s))ds,$$
$$\int_a^x G(t)dh_c(t) = G(x)h_c(x) - \int_a^x h_c(t)dG(t) = G(x)h_c(x) - \int_0^{G(x)} h_c(G^{-1}(s))ds.$$

If $F(x) = G(x)$, it is straightforward to see that (ii) implies (i). If $F(x) > G(x)$, we rewrite $\int_a^x G(t)dh_c(t) = \left(F(x)h_c(x) - \int_0^{F(x)} h_c(G^{-1}(s))ds\right) + \int_{G(x)}^{F(x)} h_c(G^{-1}(s))ds - [F(x) - G(x)]h_c(x)$, where the term in the first set of brackets is not smaller than $\int_a^x F(t)dh_c(t)$, and the residual term is nonnegative due to $h_c(G^{-1}(s)) \geq h_c(x)$ for $s \in [G(x), F(x)]$. If $F(x) < G(x)$, we rewrite $\int_a^x G(t)dh_c(t) = \left(F(x)h_c(x) - \int_0^{F(x)} h_c(G^{-1}(s))ds\right) + [G(x) - F(x)]h_c(x) - \int_{F(x)}^{G(x)} h_c(G^{-1}(s))$, where the term in the first set of brackets is not smaller than $\int_a^x F(t)dh_c(t)$, and the residual term is nonnegative due to $h_c(x) \geq h_c(G^{-1}(s))$ for $s \in [F(x), G(x)]$.

Summing up the above discussions, we obtain that (ii) implies (i). $\square$

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