Comparative Precautionary Saving under Higher-Order Risk and Recursive Utility

AJ A. Bostian Christoph Heinzel[†][*]

August 20, 2017

Abstract

Despite the importance of precautionary saving for life-cycle consumption, measuring and comparing the strength of precautionary saving motives have been conceptually restricted, under expected utility, to reactions to increases in income risk, also of higher orders, and, under recursive utility, to actuarially neutral risks added to income. This paper provides characterizations of comparative precautionary saving under recursive utility, analogous to Ross's (1981) comparative risk aversion, for increases in income risk and increases in risk on the saving return, also of higher orders. The characterizations involve comparisons in terms of precautionary premia. In addition, we show for these risk situations how the preferences underpinning precautionary saving can be measured and compared using preference coefficients.

Keywords: precautionary saving, prudence, recursive utility, higher-order risk, precautionary premium, preference coefficient

JEL classification: D91, D81

[†]Bostian: University of Tampere, Finland, aj.bostian@uta.fi. Heinzel: INRA, SMART-LERECO, AGROCAMPUS, 35000, Rennes, France, Christoph.Heinzel@inra.fr.

[*]We are grateful for valuable comments by Arthur Charpentier, Louis Eeckhoudt, Olivier l'Haridon, and Debrah Meloso; participants in the 2016 International Conference on Economic and Financial Risks in Niort; and the CEAR/MRIC Behavioral Insurance Workshop in Munich. This research was partially supported by the European Union's Seventh Framework Programme FP7/2007-2011, Grant Agreement no. 290693 (FOODSECURE).

1 Introduction

While the importance of precautionary saving for life-cycle consumption is well established, its extent and drivers are still subject to debates (e.g., Gourinchas and Parker 2001, Attanasio and Weber 2010, Choi et al. 2017, Lugilde et al. 2017). A particular lacuna concerns the measurement and comparison of the preferences which underpin precautionary saving. Such preference estimates are relevant for predictions of precautionary behaviors in fields as varied as the design of social-security and tax systems (Mirrlees et al. 2010), social discounting (Gollier 2013), and asset allocation (Gomes and Michaelides 2005).

To capture the strength of the precautionary-saving motive Kimball (1990) introduces the precautionary premium and coefficients of prudence.[1] But, akin to the Arrow-Pratt risk premium and coefficients of risk aversion in static choice, the Kimball measures are only formulated for small additive risks on future wealth and are not well-suited to analyze the preferences which underlie reactions to changes in risk (Kihlstrom et al. 1981, Ross 1981), multiplicative risks (Capéraà and Eeckhoudt 1975, Briys et al. 1989), or higher-order aspects of risk (Chiu 2005, Denuit and Eeckhoudt 2010b).

Eeckhoudt and Schlesinger (2008) (ES) determine the conditions on preferences for positive saving responses to higher-order increases in income or return risk, and argue for the potential behavioral relevance of this wide set of stochastic events. Motivated by ES, Liu (2014) generalizes Kimball's precautionary-premium analysis to higher-order increases in income risk using the higher-order extension of Ross's approach to comparative risk aversion of Liu and Meyer (2013b). Liu thereby provides a systematic framework to compare the precautionary-saving motives of agents in such risk situations, but he does not treat the more involved case with return risk nor comparisons using preference coefficients.

All aforementioned studies on precautionary saving (apart from Choi et al. and Gomes and Michaelides) focus on the traditional univariate time-additive expected utility (EU) model. This model has the unfortunate property of equating risk and intertemporal preferences, precluding to study the potential distinct roles of these two preference di-

[1] Kimball describes the precautionary-saving motive as the "propensity to prepare and forearm oneself" to better cope with future risk. We refer to both absolute and relative coefficients below.

mensions. Many empirical studies indeed indicate that the preferences from the risk and intertemporal domains are, in general, not linked in the specific manner assumed by EU. Recursive utility (RU), conceived by Kreps and Porteus (1978) and Selden (1978) and popularized by Epstein and Zin (1989, 1991) and Weil (1990) (EZW), disentangles these roles, keeping EU as a special case. RU has become a standard model in settings with an intertemporal smoothing variable.[2]

Interestingly, only a small part of the literature using RU deals with precautionary saving.[3] Bostian and Heinzel (2017) extend ES's analysis to RU and study the roles of the different preference features qualitatively and quantitatively. While risk aversion and the appropriate higher-order risk preferences are crucial for precautionary saving to arise, lower-order effects, and with it consumption smoothing, tend to have a dominating influence on total saving, at the expense of the proportion of precautionary saving. In that vein, the relative risk aversion (RRA) coefficient in the intertemporal EU model primarily conveys information about consumption-smoothing, and not risk preferences. Given the essential role of each of these preference features, predictions of precautionary saving should involve appropriate estimates of all of them.[4]

To our knowledge, the only constructive contribution to the measurement and comparison of precautionary-saving motives under RU is Kimball and Weil (2009) (KW), who adapt Kimball's analysis to RU. KW's precautionary premium and prudence coefficients show the simultaneous action of intertemporal and risk preferences, but their scope is restricted like Kimball's. To date, no systematic approach is available to compare the motives which underlie under RU precautionary-saving reactions to risk increases, return risk, or higher-order risk aspects.

This paper provides such an approach in three steps. First, we extend Liu's precautionary-premium analysis under income risk to RU. According to the two risk-impact channels

[2]For example, in macroeconomics and finance RU has much contributed to explanations of the equity-premium and risk-free-rate puzzles (e.g., Hall 1988, Weil 1989, Epstein and Zin 1991, Tallarini 2000, Bansal and Yaron 2004, Barro 2009, Binsbergen et al. 2012, Martin 2013, Epstein et al. 2014).

[3]Apart from the papers mentioned in the text, notable contributions include Selden (1979), Weil (1993), Nocetti and Smith (2010, 2011a,b), Bommier et al. (2012), and Wang et al. (2016).

[4]Studies measuring the precautionary-saving motive from field data concentrate thus far on second-order income risk and EU, with inconclusive results on its prevalence and strength (e.g., Dynan 1993, Merrigan and Normandin 1996, Eisenhauer 2000, Ventura and Eisenhauer 2006, Lee and Sawada 2007).

on saving under RU, the characterizations need to control for risk preferences in two ways. Similar to KW, sufficient conditions arise for a uniform ordering of the precautionary premia of two RU agents, given certain constellations of their second-order risk and intertemporal preferences.

In a second step, we define a precautionary premium for increases in return risk, adapting the multiplicative risk-premium concept of Eeckhoudt and Schlesinger (2009). Under return risk, risk exposure is directly endogenous to saving. The saving reaction is positive or negative depending on whether a positive precautionary or a negative substitution effect dominates. The precautionary premium has the sign of the saving reaction. The characterizations of comparative precautionary saving arise similarly to income risk, but now have two formulations depending on the sign of the saving reactions. Because the precautionary premium only reflects the net of the two effects, its merit as a measure of the preferences underpinning precautionary saving is limited.

We finally define a class of preference coefficients building on Liu and Meyer (2013a,b) which enable us to alternatively represent comparative precautionary saving in the same conceptual framework for income risk and return risk.

Section 2 introduces the decision framework for the analysis. We provide the three steps of our analysis in Sections 3 to 5.

2 Decision Framework

Following KW, we start from a two-period consumption/saving model under RU. The agent chooses a saving amount s out of first-period income y_1 that maximizes the intertemporal utility objective

$$u\left(y_1 - s\right) + \beta u\left(CE(\tilde{y}_2 + s\tilde{R})\right) \tag{1}$$

where u is the felicity function capturing the preference for smooth consumption, and β is the agent's utility discount factor. Second-period consumption $\tilde{c}_2$ is funded by income $\tilde{y}_2$ and saving with gross return $\tilde{R}$. Risk may enter through either $\tilde{y}_2$ or $\tilde{R}$, and a tilde indicates that a variable is risky.[5] The certainty equivalent $CE(\tilde{c}_2)$ ranks future consumption according to the agent's risk preference ψ:

$$CE(\tilde{y}_2 + s\tilde{R}) \equiv \psi^{-1}\left(E\left[\psi\left(\tilde{y}_2 + s\tilde{R}\right)\right]\right) \tag{2}$$

Unlike u, ψ is a von Neumann-Morgenstern utility function (Selden 1978). Special cases of (1) are EU (for $\psi = u$) and EZW (for constant RRA and EIS).

According to the first-order optimality condition

$$u'(c_1) = \beta u'(CE(\tilde{c}_2))\, CE'(\tilde{c}_2) \tag{3}$$

the agent saves until the marginal utility from foregoing consumption in period 1 (i.e., saving a marginal amount) is equal to the discounted marginal utility from consuming instead in period 2. With some abuse of notation for compactness, the quantity CE' in (3) is the full derivative of certainty equivalent (2) with respect to saving:

$$CE'(\tilde{c}_2) \equiv \frac{dCE(\tilde{c}_2)}{ds} = \frac{E[\psi'(\tilde{c}_2)\tilde{R}]}{\psi'(CE(\tilde{c}_2))} \tag{4}$$

For the second-order condition to this problem to be fulfilled, some atypical restrictions must be placed on u and ψ. Both functions must be increasing and concave, but ψ^{-1} must also be concave in s. This last requirement is fulfilled if ψ's absolute risk tolerance – the inverse of absolute risk aversion – is concave (KW, Gollier 2001). Under these assumptions, which we maintain throughout, $CE''(\tilde{c}_2) < 0$,[6] so that:

$$u''(c_1) + \beta\left[u''(CE(\tilde{c}_2))\,[CE'(\tilde{c}_2)]^2 + u'(CE(\tilde{c}_2))\, CE''(\tilde{c}_2)\right] < 0$$

In addition, we assume that ψ exhibits decreasing absolute risk aversion (DARA).

When analytically more convenient, we use, like KW, a representation of utility ob-

[5] We consider either income risk or return risk. If $\tilde{y}_2$ and $\tilde{R}$ carry a tilde, the notation applies to each of the two cases.

[6] The proof in KW (Appendix A) holds analogously under return risk.

jective (1) which involves the Kreps-Porteus (KP) operator $\phi(.) \equiv u(\psi^{-1}(.))$,

$$u\left(y_1 - s\right) + \beta\phi(E\psi(\tilde{y}_2 + s\tilde{R}))$$

The two representations are equivalent if ψ is a continuous and monotonically increasing function, making CE in (2) well-defined when $E\psi(\tilde{c}_2)$ is well-defined. The curvature of ϕ depends on the relation of second-order intertemporal and risk preferences, and relates to a preference for the timing of risk resolution: if $\phi(.)$ is convex (concave) or, equivalently, $-u''/u' < (>) - \psi''/\psi'$, the agent has a preference for early (late) risk resolution. Which preference prevails is an empirical question which we do not address in this paper.

ES for EU and Bostian and Heinzel for RU as in (1) derive conditions on preferences for saving to increase in response to deteriorations in n^{th}-order stochastic dominance (NSD). Decisive for their proofs is Lemma 1, which we adopt from ES:[7]

Lemma 1 (NSD Equivalence) *Let $\tilde{z}_l$ and $\tilde{z}_h$ be two random variables with identical real bounded support. The following two statements are equivalent:*

1. *$\tilde{z}_l$ dominates $\tilde{z}_h$ via NSD.*

2. *$Ef'(\tilde{z}_l) \leq Ef'(\tilde{z}_h)$ for any function $f(t)$ with $\operatorname{sgn}\left[f^{(n)}(t)\right] = (-1)^n$ for all $n = 1, 2, \ldots, N+1$.*

Lemma 1 helps to link risk increases and the higher-order preferences they activate. Under EU, for $f = u_{eu}$, the preference conditions for saving increases derive directly from this lemma, as we explain below.

Under RU, its scope is more restricted. Inserting the expression on the right in (4) into first-order condition (3) shows that risk affects future marginal RU via two channels, expected marginal utility $E_1[\psi'(\tilde{c}_2)\tilde{R}]$ and the certainty equivalent $CE(\tilde{c}_2)$:

$$u'\left(CE(\tilde{c}_2)\right) CE'(\tilde{c}_2) = \frac{u'\left(CE(\tilde{c}_2)\right)}{\psi'\left(CE(\tilde{c}_2)\right)} E[\psi'(\tilde{c}_2)\tilde{R}] = \phi'(E\psi(\tilde{c}_2)) E[\psi'(\tilde{c}_2)\tilde{R}] \tag{5}$$

$E[\psi'(\tilde{c}_2)\tilde{R}]$ has a direct positive influence in (5), whereas the impact of CE is positive

[7] Where unambiguous, $f^{(n)}(t) \equiv \frac{\partial^n f(t)}{\partial t^n}$ denotes an n^{th} derivative.

(negative) depending on whether $\phi(.)$ is convex (concave). Lemma 1 provides here for $f = \psi$ the conditions for uniform risk effects on $E[\psi'(\tilde{c}_2)\tilde{R}]$.

For the analysis in this paper, we refer to a special case of NSD in form of the nth-degree first-ℓ-moments-preserving stochastic dominance (n-ℓ-MPSD) order from Liu:

Definition 1 (nth-degree ℓ-first-Moments-Preserving Stochastic Dominance) *For any integer ℓ with $1 \leq \ell \leq n-1$, $\tilde{x}_l$ dominates $\tilde{x}_h$ by n-ℓ-MPSD if and only if $\tilde{x}_l \succeq_{nSD} \tilde{x}_h$ and $E\left(\tilde{x}_l^j\right) = E\left(\tilde{x}_h^j\right)$ for $j = 1, \ldots, \ell$.*

In Definition 1, n indicates the stochastic-dominance order used in comparing the risky variables $\tilde{x}_l$ (the "lower risk" lottery) and $\tilde{x}_h$ (the "higher risk" lottery), and ℓ denotes how many of their lower moments are identical. We use the n-ℓ-MPSD order because of its flexible intermediate position between n^{th}-degree Ekern risk increases (for $\ell = n-1$) and the nth-degree mean-preserving stochastic dominance of Denuit and Eeckhoudt (2013) (for $\ell = 1$), which it covers both as special cases.[8]

The conditions on preferences for saving increases in response to risk increases differ between EU and RU for income and return risk. For a deterioration from $\tilde{y}_{2,l}$ to $\tilde{y}_{2,h}$ in the n-ℓ-MPSD order, ES's analysis implies that an EU agent saves more if and only if, for $k = \ell+1, \ldots, n$, the sign of the $(k+1)^{th}$ EU derivative is negative (positive) for k even (odd). For example, for a mean-preserving spread, i.e., $(n, \ell) = (2, 1)$, we have the well-known condition that $u_{eu}''' \geq 0$.

The corresponding conditions in Bostian and Heinzel state preference properties guaranteeing a uniformly positive effect on future marginal RU via the two risk-impact channels. The conditions on ψ for $E\psi'(\tilde{c}_2^{y_2})$ to increase are analogous to ES's on u_{eu}. The relation between the $\tilde{y}_2$ deterioration and CE derives when inserting the associated risk premium $\pi_\psi^{y_2}$ in the rewritten equation (2):[9]

$$\psi(CE(\tilde{y}_{2,h} + sR)) = E\psi\left(\tilde{y}_{2,l} + sR - \pi_\psi^{y_2}\right) = E\psi\left(\tilde{y}_{2,h} + sR\right) \quad (6)$$

[8]For example, mean-preserving spreads (Rothschild and Stiglitz 1970) are second-degree Ekern deteriorations and 2-1-MPSD shifts; increases in downside risk (Menezes et al. 1980) are third-degree Ekern deteriorations and 3-2-MPSD shifts; and increases in outer risk (Menezes and Wang 2005) are fourth-degree Ekern deteriorations and 4-3-MPSD shifts.

[9]Denuit and Eeckhoudt (2010a) call a premium like $\pi_\psi^{y_2}$ "partial" risk premium, because it only represents an agent's willingness-to-pay to avoid the given risk increase.

$\pi_\psi^{y_2}$ increases if and only if $\text{sgn}\left[\psi^{(k)}(.)\right] = (-1)^{k+1}$ for all $k = \ell+1, \ldots, n$ (e.g., Eeckhoudt and Schlesinger 2006). For $(n, \ell) = (2, 1)$, hence, $\psi''' > 0$ ensures a positive effect on $E\psi'(\tilde{c}_2^{y_2})$, and $\psi'' > 0$ a negative effect on CE. Intertemporal preferences only come into play in relation to the first-order effect from CE in (5). Saving increases if and only if the combined impact from the two channels on future marginal RU is positive.

The saving reaction to a return-risk increase depends on the interplay of a positive precautionary and a negative substitution effect. For a deterioration from $\tilde{R}_l$ to $\tilde{R}_h$ in the n-ℓ-MPSD order, ES's EU analysis implies that the precautionary effect obtains, and saving increases, if and only if the coefficients of partial $(k+1)^{th}$-order risk aversion exceed k for $k = \ell+1, \ldots, n$:[10]

$$-\frac{u_{eu}^{(k+1)}(y_2 + sR)}{u_{eu}^{(k)}(y_2 + sR)}\, sR \;\geq\; k \tag{7}$$

For a mean-preserving spread, the coefficient involving the third and second utility derivatives has to exceed $n = 2$: $-\frac{u_{eu}'''(y_2+sR)}{u_{eu}''(y_2+sR)}\, sR \geq 2$.

For RU, the structure of the preference conditions for saving increases implied by Bostian and Heinzel is similar to the income-risk case. For $E[\psi'(\tilde{c}_2^R)\tilde{R}]$ to increase, conditions (7) have to hold, restated for ψ instead of u_{eu}. To relate the $\tilde{R}$ deterioration and CE, the authors refer to the multiplicative risk premium π^R of Eeckhoudt and Schlesinger (2009), which arises by comparing the EU of future consumption,

$$\psi(CE(y_2 + s\tilde{R}_h)) = E\psi(y_2 + s(\tilde{R}_l - \pi^R)) = E\psi(y_2 + s\tilde{R}_h) \tag{8}$$

π^R measures the proportion of saving such that the product with saving $s\pi^R$ is equal to the maximum amount of future consumption the agent is willing to forgo to avoid the risk increase. Similar to π^{y_2}, π^R is positive (zero, negative) under risk aversion (risk neutrality, risk love), and it increases if and only if $\text{sgn}\left[\psi^{(k)}(.)\right] = (-1)^{k+1}$ for all $k = \ell+1, \ldots, n$. For $(n, \ell) = (2, 1)$, $E[\psi'(\tilde{c}_2^R)\tilde{R}]$ increases, hence, when $-\frac{\psi'''(y_2+sR)}{\psi''(y_2+sR)}\, sR \geq 2$, and $\psi'' > 0$ ensures a negative effect on CE. Intertemporal preferences again steer the first-order effect from CE in (5), and for saving to increase the combined impact from the two channels on

[10] ES state these conditions for suppressed risk and $y_2 = 0$.

future marginal RU must be positive. Importantly, the analysis in Bostian and Heinzel suggests that agents tend to substitute away from return risk, instead of increasing saving (and thereby risk exposure). A negative impact seems thus empirically relevant. We particularly account for this case below.

Kimball introduces two concepts to measure an EU agent's precautionary-saving motive in the face of a small zero-mean risk added to certain exogenous future income. The precautionary premium θ_{eu} captures the agent's maximum willingness to save to confront this risk. It derives by equating future marginal utility evaluated at expected future consumption minus θ_{eu} and the expected future marginal utility under risk:

$$u'_{eu}(E\tilde{c}_2^{y_2} - \theta_{eu}) = Eu'_{eu}(\tilde{c}_2^{y_2})$$

θ_{eu} is thus the safe reduction in $\tilde{y}_2$ that generates the same effect on saving as adding the small zero-mean risk to $\tilde{y}_2$. In addition, the coefficient of relative prudence measures the intensity of the precautionary-saving motive,

$$RP_{eu}(c_2^{y_2}) = -\frac{c_2^{y_2} u'''_{eu}(c_2^{y_2})}{u''_{eu}(c_2^{y_2})} \tag{9}$$

KW adapt these concepts to RU, where the precautionary premium θ derives from:

$$u'(E\tilde{c}_2^{y_2} - \theta) = u'(CE(\tilde{c}_2^{y_2}))CE'(\tilde{c}_2^{y_2}) \tag{10a}$$

and the coefficient of relative prudence arises as

$$RP\,(c_2^{y_2}) = RRA_\psi(c_2^{y_2})\,[1 + \varepsilon_\psi(c_2^{y_2})\,EIS_u(c_2^{y_2})] \tag{10b}$$

Coefficient (10b) gives an insight into the interplay of the different preference dimensions. It predicts prudence to increase with risk aversion, the elasticity of absolute risk tolerance $\varepsilon_\psi = RP_\psi - RRA_\psi$, and the elasticity of intertemporal substitution (EIS_u).

In contrast to the coefficients by Arrow-Pratt and, in part, Kimball, the KW coefficient has, to our knowledge, not been examined regarding its potential application in the broad set of risk situations in ES. Because Kimball and KW determine their coefficients with a procedure similar to the Arrow-Pratt approximation, these results are still informative for the KW coefficient. Two main extensions in ES of the standard risk setting regard risk increases in situations where risk is already present, and higher-order risk.[11] Regarding the first, for example, Kihlstrom et al. (1981) and Ross (1981) find that conditions stronger than Arrow-Pratt's are needed to compare two agents' risk aversion. Chiu (2005), moreover, determines the conditions under which the analog of Kimball's prudence coefficient in static choice can represent attitudes towards increases in downside risk. Denuit and Eeckhoudt (2010b) extend this analysis to nth-order Ekern risk increases.

Unlike income risk, return risk generates a multiplicative risk exposure that is directly scaled by saving. Unfortunately, the Arrow-Pratt risk premium is not well defined for multiplicative risks (Capéraà and Eeckhoudt 1975, Briys et al. 1989). For example, it need neither be positive for risk averters nor increase with mean-preserving spreads. Also the premium measures for return risk suggested by Drèze and Modigliani (1972) (for EU) and Langlais (for RU) suffer from these issues. Therefore, we build in this paper on the multiplicative risk-premium concept as in (8).

To compare attitudes towards risk, we adopt, similar to Liu, the Liu and Meyer (2013b) concept of $(n/m)^{th}$-degree Ross more risk aversion. For its definition, consider the class $\mathcal{U}^D_{k-cv}$ of all k-concave functions f on D, formally the set $\{f|(-1)^k f^{(k)}(z) \leq 0$ for some $k \in \mathbb{N}$ and all $z \in D \subseteq \mathbb{R}\}$.

Definition 2 ($(n/m)^{th}$-Degree Ross More Risk Aversion) *For two utility functions $\psi_u(x), \psi_v(x) \in \mathcal{U}^{D_x}_{n-cv} \times \mathcal{U}^{D_x}_{m-cv}$, $n > m$, $\psi_u(x)$ is $(n/m)^{th}$-degree Ross more risk-averse than $\psi_v(x)$ if and only if there exists a $\lambda > 0$ such that*

$$\frac{\psi_u^{(n)}(x_a)}{\psi_v^{(n)}(x_a)} \geq \lambda \geq \frac{\psi_u^{(m)}(x_b)}{\psi_v^{(m)}(x_b)} \quad \textit{for all} \;\; x_a, x_b \in D_x \subseteq \mathbb{R}_0^+ \tag{11}$$

Definition 2 compares the risk preferences of agents u and v. In general, intertemporal preferences u and v and risk preferences ψ_u and ψ_v differ between the two agents. For notational convenience, we drop the u or v subscripts at CE, ψ, and ϕ whenever these

[11] Eeckhoudt (2014) provides a summary of the related literature.

functions are clearly assigned to u or v in an expression.

3 Increases in Income Risk

At the center of the comparisons of precautionary-saving attitudes for increases in income risk is a characterization in terms of the RU precautionary premium θ^{y_2}. We define this premium as the solution to either of the two equivalent equations:

$$u'(CE(\tilde{y}_{2,l} + sR - \theta_u^{y_2}))CE'(\tilde{y}_{2,l} + sR - \theta_u^{y_2}) = u'(CE(\tilde{y}_{2,h} + sR))CE'(\tilde{y}_{2,h} + sR) \quad (12a)$$

$$\phi_u'(E\psi(\tilde{y}_{2,l} + sR - \theta_u^{y_2}))E\psi'(\tilde{y}_{2,l} + sR - \theta_u^{y_2}) = \phi_u'(E\psi(\tilde{y}_{2,h} + sR))E\psi'(\tilde{y}_{2,h} + sR) \quad (12b)$$

In contrast to KW's premium in (10a), which refers to the entire risk, θ^{y_2} is the safe reduction in $\tilde{y}_{2,l}$ that has the same effect on saving as the change from $\tilde{y}_{2,l}$ to $\tilde{y}_{2,h}$.

While $\pi_\psi^{y_2}$ from (6) captures the strength of the risk preferences which mediate the risk effect via CE, we can measure the strength of the risk impact via $E\psi'(\tilde{c}_2^{y_2})$ at the von Neumann-Morgenstern (vNM) precautionary premium $\theta_\psi^{y_2}$ (KW, Liu). It derives from

$$E\psi'(\tilde{y}_{2,l} + sR - \theta_\psi^{y_2}) = E\psi'(\tilde{y}_{2,h} + sR) \quad (13)$$

Inserting $\pi_\psi^{y_2}$ and $\theta_\psi^{y_2}$ into (12b) provides a quantitative link between the risk attitudes which control the two risk effects and the precautionary-saving motive:[12]

$$\phi_u'(E\psi(\tilde{y}_{2,l} + sR - \pi_\psi^{y_2}))E\psi'(\tilde{y}_{2,l} + sR - \theta_\psi^{y_2}) = \phi_u'(E\psi(\tilde{y}_{2,h} + sR))E\psi'(\tilde{y}_{2,h} + sR) \quad (14)$$

In line with the above explanations, θ^{y_2} increases with $\theta_\psi^{y_2}$. However, the effect of $\pi_\psi^{y_2}$ is ambiguous: θ^{y_2} increases (decreases) with $\pi_\psi^{y_2}$ when $\phi_u(.)$ is concave (convex).

Like in KW, there is a positive relation between the risk and vNM precautionary premia, as implied by the following two inequalities. Starting from (6), we have thanks

[12] KW derive similar relations among the three premia in their context referring to compensating premia. We focus instead on equivalent premia in this paper.

to DARA of ψ that for all $\delta \geq 0$

$$E\psi(\tilde{y}_{2,l} + sR - \pi_\psi^{y_2} + \delta) \leq E\psi(\tilde{y}_{2,h} + sR + \delta)$$

so that at the margin, for $\delta = 0$,

$$E\psi'(\tilde{y}_{2,l} + sR - \pi_\psi^{y_2}) \leq E\psi'(\tilde{y}_{2,h} + sR)$$

Comparing the latter inequality to (13) (along with $\psi'' \leq 0$) implies that $\theta_\psi^{y_2} \geq \pi_\psi^{y_2}$. Similar to KW, it also holds that $\theta^{y_2} \geq \pi^{y_2}$, and $\theta^{y_2} \leq (\geq)\ \theta_\psi^{y_2}$ if ϕ is concave (convex).[13] While these results provide insights in the relations among the different preference elements at the individual level, they are, because of the two risk-impact channel under RU, not directly helpful to compare the precautionary-saving motives of two agents.

Theorem 1 compares, in analogy to Liu under EU, the saving attitudes of two RU agents u and v who have the same saving level under $\tilde{y}_{2,l}$. For uniform effects across the two risk-impact channels, we need to control for intertemporal and risk preferences separately, and for the relations of risk preferences in each channel in turn. Only three, out of six possible, curvature constellations between ϕ_u and ϕ_v provide, given appropriate relations of risk preferences, sufficient conditions for $\theta_u^{y_2} \geq \theta_v^{y_2}$. Due to the two risk-impact channels, we have only sufficient conditions for the order of the RU precautionary premia, but not necessity (similar to KW, Propositions 5 and 6).

Theorem 1 *Suppose $n \geq 2$ and $1 \leq \ell \leq n-1$, and consider two RU agents u, v with risk preferences ψ_u, ψ_v in the class $\mathcal{U}_{k-cv}^{D_{c^{y_2}}}$ of all k-concave functions for $k = 1, 2$ and $\ell+1, \ldots, n+1$ and twice differentiable KP operators ϕ_u, ϕ_v with $\phi_u', \phi_v' \geq 0$, who save the same amount under the reference income $\tilde{y}_{2,l}$. Then, conditions (i)–(iii) are equivalent*

[13] $\pi^{y_2} \leq \theta^{y_2}$ follows from (14) and $\theta_\psi^{y_2} \geq \pi_\psi^{y_2}$, because, due to $CE'' < 0$, the left-hand side of (12a), and thus that of (12b), depend negatively on θ^{y_2}. For $\theta^{y_2} \leq (\geq)\ \theta_\psi^{y_2}$, note that from (13), $\theta_\psi^{y_2} \geq \pi_\psi^{y_2}$, (6), and (12b) together with its negative dependence on θ^{y_2}, we have, when ϕ is concave (convex), that:

$$\begin{aligned}\phi_u'(E\psi(\tilde{y}_{2,l} + sR - \theta^{y_2}))E\psi'(\tilde{y}_{2,l} + sR - \theta^{y_2}) &= \phi_u'(E\psi(\tilde{y}_{2,l} + sR - \theta^{y_2}))E\psi'(\tilde{y}_{2,h} + sR) \\ &\geq (\leq)\ \phi_u'(E\psi(\tilde{y}_{2,l} + sR - \pi^{y_2}))E\psi'(\tilde{y}_{2,h} + sR) \\ &= \phi_u'(E\psi(\tilde{y}_{2,h} + sR))E\psi'(\tilde{y}_{2,h} + sR).\end{aligned}$$

and imply, if $\phi_u'' \leq \phi_v'' \leq 0$, condition (iv):

(i) ψ_u *is* $(k/1)^{th}$*- and* $((k+1)/2)^{th}$*-degree Ross more risk-averse than* ψ_v *for all* $k = \ell+1, \ldots, n$.

(ii) For $\lambda_\gamma, \lambda_\eta > 0$, $\gamma(c^{y_2})$ *with* $\gamma'(c^{y_2}) \leq 0$, $\gamma(c^{y_2}) \in \mathcal{U}_{k-cv}^{D_{c^{y_2}}}$, *and* $\eta'(c^{y_2})$ *with* $\eta''(c^{y_2}) \geq 0$, $\eta'(c^{y_2}) \in \mathcal{U}_{(k+1)-cv}^{D_{c^{y_2}}}$, *each for* $k = \ell+1, \ldots, n$ *and all* $c^{y_2} \in D_{c^{y_2}} \subset \mathbb{R}_0^+$, *it is*

$$\psi_u(c^{y_2}) = \lambda_\gamma \psi_v(c^{y_2}) + \gamma(c^{y_2}) \tag{15a}$$

$$\psi_u'(c^{y_2}) = \lambda_\eta \psi_v'(c^{y_2}) + \eta'(c^{y_2}) \tag{15b}$$

(iii) $\pi_{\psi_u}^{y_2} \geq \pi_{\psi_v}^{y_2}$ *and* $\theta_{\psi_u}^{y_2} \geq \theta_{\psi_v}^{y_2}$ *for all* $\tilde{y}_{2,l}, \tilde{y}_{2,h}$ *with* $\tilde{y}_{2,l} \succeq_{n-\ell-MPSD} \tilde{y}_{2,h}$, $\pi_u^{y_2}$ *and* $\theta_{\psi_u}^{y_2}$ *as defined, respectively, in* (6) *and* (13) *for* u, *and analogously* $\pi_v^{y_2}, \theta_{\psi_v}^{y_2}$ *for agent* v.

(iv) $\theta_u^{y_2} \geq \theta_v^{y_2}$ *for all* $\tilde{y}_{2,l}, \tilde{y}_{2,h}$ *with* $\tilde{y}_{2,l} \succeq_{n-\ell-MPSD} \tilde{y}_{2,h}$, $\theta_u^{y_2}$ *as defined in* (12) *for* u, *and analogously* $\theta_v^{y_2}$ *for agent* v.

The equivalences hold similarly and imply, if $0 \leq \phi_u'' \leq \phi_v''$, *condition (iv), given the following adjustments,*

- *in (i):* ψ_v *is* $(k/1)^{th}$*-degree Ross more risk-averse than* ψ_u, *instead of the reverse;*
- *in (ii):* (15a) *is replaced by* $\psi_v(c^{y_2}) = \lambda_\gamma \psi_u(c^{y_2}) + \gamma(c^{y_2})$;
- *in (iii):* $\pi_{\psi_v}^{y_2} \geq \pi_{\psi_u}^{y_2}$, *instead of the reverse.*

If $\phi_u'' \leq 0 \leq \phi_v''$, *both sets of conditions (i)–(iii) imply condition (iv).*

The equivalences among conditions (i), (ii), and (iii), and similarly those among the three adjusted conditions, follow as in Liu. The conclusion for $\theta_u^{y_2} \geq \theta_v^{y_2}$ in condition (iv) is immediate from (14) in each case.[14]

[14] In all other cases, the effect from the CE channel on the order of $\theta_u^{y_2}$ and $\theta_v^{y_2}$ is either ambiguous at the given level of generality ($\phi_u'' \leq \phi_v'' \leq 0$ or $0 \leq \phi_v'' \leq \phi_u''$ combined with the adjusted conditions; $0 \leq \phi_u'' \leq \phi_v''$ or $\phi_v'' \leq \phi_u'' \leq 0$ combined with condition (i)–(iii)), or there is a stronger positive influence on $\theta_v^{y_2}$ than on $\theta_u^{y_2}$ ($\phi_v'' \leq \phi_u'' \leq 0$ combined with the adjusted conditions; $0 \leq \phi_v'' \leq \phi_u''$ combined with condition (i)–(iii); $\phi_v'' \leq 0 \leq \phi_u''$ combined with either set of conditions).

While the conditions on risk preferences in the $E\psi'(\tilde{c}^{y_2})$ channel mirror those on intertemporal EU in Liu, the CE channel reveals an interaction between intertemporal and higher-order risk preferences absent under EU: if ϕ_u and ϕ_u are concave $\phi_u'' \leq \phi_v'' \leq 0$ goes with $\pi_{\psi_u}^{y_2} \geq \pi_{\psi_v}^{y_2}$, but if ϕ_u and ϕ_u are convex $0 \leq \phi_u'' \leq \phi_v''$ goes with $\pi_{\psi_u}^{y_2} \leq \pi_{\psi_v}^{y_2}$ to guarantee higher precautionary saving from this channel. That is, under RU an agent's higher risk aversion may impede the establishment of unambiguous conditions for this agent's stronger precautionary-saving motive. This illustrates that appropriately controlling for the relations between intertemporal and risk preferences within and between agents is crucial for valid comparisons of precautionary-saving motives under RU.

4 Increases in Return Risk

We derive the RU precautionary premium θ^R for return-risk increases adapting the multiplicative risk-premium concept (8). θ^R derives from either of the equations:

$$u'(CE(y_2 + s(\tilde{R}_l - \theta_u^R)))CE'(y_2 + s(\tilde{R}_l - \theta_u^R)) = u'(CE(y_2 + s\tilde{R}_h))CE'(y_2 + s\tilde{R}_h) \tag{16a}$$

$$\begin{aligned} &\phi_u'(E\psi(y_2 + s(\tilde{R}_l - \theta_u^R)))E[\psi'(y_2 + s(\tilde{R}_l - \theta_u^R))\tilde{R}_l] \\ &\qquad = \phi_u'(E\psi(y_2 + s\tilde{R}_h))E[\psi'(y_2 + s\tilde{R}_h)\tilde{R}_h] \end{aligned} \tag{16b}$$

θ^R is the proportion of saving, so that the amount $s\theta^R$ is the safe variation in $\tilde{c}_2^R$ that generates the same effect on saving as the deterioration from $\tilde{R}_l$ to $\tilde{R}_h$. Importantly, the sign of θ^R depends on the sign of the saving response to return risk: θ^R is positive (zero, negative) if saving increases (stays constant, falls). To see this, note that because future marginal RU decreases in saving, the marginal RU premium

$$\begin{aligned} &u'(CE(y_2 + s\tilde{R}_h))CE'(y_2 + s\tilde{R}_h) - u'(CE(y_2 + s\tilde{R}_l))CE'(y_2 + s\tilde{R}_l) \; = \\ &\phi_u'(E\psi(y_2 + s\tilde{R}_h))E[\psi'(y_2 + s\tilde{R}_h)\tilde{R}_h] - \phi_u'(E\psi(y_2 + s\tilde{R}_l))E[\psi'(y_2 + s\tilde{R}_l)\tilde{R}_l] \end{aligned}$$

is positive (negative) if and only if the precautionary (substitution) effect dominates, so that saving under $\tilde{R}_h$, s^{*h}, is higher (lower) than saving under $\tilde{R}_l$, s^{*l}. If the two effects cancel, the marginal RU premium is zero. Because $CE'' < 0$, future marginal RU on the left-hand side of (16a), and thus (16b), depends positively on θ^R, so that we have

$$s^{*h} \gtreqless s^{*l} \;\Leftrightarrow\; \theta^R \gtreqless 0$$

Moreover, θ^R increases in absolute terms with the strength of the saving reaction.

We can measure the strength of the risk preferences which mediate the risk effect via CE at π_ψ^R from (8). The risk impact via $E[\psi'(\tilde{c}_2^R)\tilde{R}]$ in (16b) derives from Lemma 1 stated for ψ instead of u_{eu}. The vNM precautionary premium θ_ψ^R, deriving from

$$E[\psi'(y_2 + s(\tilde{R}_l - \theta_\psi^R))\tilde{R}_l] = E[\psi'(y_2 + s\tilde{R}_h)\tilde{R}_h] \tag{17}$$

then, measures the strength of the risk impact on saving via $E[\psi'(\tilde{c}_2^R)\tilde{R}]$. Similar to θ^R, the sign of θ_ψ^R can be positive or negative. Because equation (17) is, like equations (16), evaluated at s^{*h} and its left-hand side depends positively on θ_ψ^R, it holds that

$$s^{*h} \gtreqless s^{*l} \;\Leftrightarrow\; \theta_\psi^R \gtreqless 0$$

Inserting π_ψ^R and θ_ψ^R into (16b) links again the risk attitudes which control the two risk effects and the precautionary-saving motive quantitatively:

$$\phi_u'(E\psi(y_2 + s(\tilde{R}_l - \pi_\psi^R)))E[\psi'(y_2 + s(\tilde{R}_l - \theta_\psi^R))\tilde{R}_l] = \phi_u'(E\psi(y_2 + s\tilde{R}_h))E[\psi'(y_2 + s\tilde{R}_h)\tilde{R}_h] \tag{18}$$

While there is an immediate positive relation between θ^R and θ_ψ^R, the effect of π_ψ^R is, as before, ambiguous: θ^R increases (decreases) with π_ψ^R when $\phi_u(.)$ is concave (convex).

The comparisons of precautionary behavior under return risk proceed much like under income risk, only that Theorem 2 has two formulations depending on whether agents u and v both increase or decrease saving in response to a given return-risk increase.[15]

[15]The cases with saving responses of different signs between agents u and v are not particularly inter-

Theorem 2 *Suppose $n \geq 2$ and $1 \leq \ell \leq n-1$, and consider two RU agents u, v with risk preferences ψ_u, ψ_v in the class $\mathcal{U}_{k-cv}^{D_{c^R}}$ of all k-concave functions for $k = 1, 2$ and $\ell+1, \ldots, n+1$ and twice differentiable KP operators ϕ_u, ϕ_v with $\phi_u', \phi_v' \geq 0$, who save the same amount under the reference return $\tilde{R}_l$ and increase [decrease] saving in response to return-risk increases in the n-ℓ-MPSD order. Then, conditions (i)–(iii) are equivalent and imply, if $\phi_u'' \leq \phi_v'' \leq 0$ $[\, 0 \leq \phi_v'' \leq \phi_u''\,]$, condition (iv):*

(i) ψ_u is $(k/1)^{th}$- and $((k+1)/2)^{th}$-degree Ross more risk-averse than ψ_v for all $k = \ell+1, \ldots, n$.

(ii) For $\lambda_\gamma, \lambda_\eta > 0$, $\gamma(c^R)$ with $\gamma'(c^R) \leq 0$, $\gamma(c^R) \in \mathcal{U}_{k-cv}^{D_{c^R}}$, and $\eta'(c^R)$ with $\eta''(c^R) \geq 0$, $\eta'(c^R) \in \mathcal{U}_{(k+1)-cv}^{D_{c^R}}$, each for $k = \ell+1, \ldots, n$ and all $c^R \in D_{c^R} \subset \mathbb{R}_0^+$, it is

$$\psi_u(c^R) = \lambda_\gamma \psi_v(c^R) + \gamma(c^R) \tag{19a}$$

$$\psi_u'(c^R) = \lambda_\eta \psi_v'(c^R) + \eta'(c^R) \tag{19b}$$

(iii) $\pi_{\psi_u}^R \geq \pi_{\psi_v}^R$ and $\theta_{\psi_u}^R \geq \theta_{\psi_v}^R \geq 0$ $\left[\, \theta_{\psi_u}^R \leq \theta_{\psi_v}^R \leq 0 \,\right]$ for all $\tilde{R}_l, \tilde{R}_h$ with $\tilde{R}_l \succeq_{n-\ell-MPSD} \tilde{R}_h$, π_u^R and $\theta_{\psi_u}^R$ as defined, respectively, in (8) and (17) for u, and analogously $\pi_v^R, \theta_{\psi_v}^R$ for agent v.

(iv) $\theta_u^R \geq \theta_v^R \geq 0$ $\left[\, \theta_u^R \leq \theta_v^R \leq 0 \,\right]$ for all $\tilde{R}_l, \tilde{R}_h$ with $\tilde{R}_l \succeq_{n-\ell-MPSD} \tilde{R}_h$, θ_u^R as defined in (16) for u, and analogously θ_v^R for agent v.

The equivalences hold similarly and imply, if $0 \leq \phi_u'' \leq \phi_v''$ $[\, \phi_v'' \leq \phi_u'' \leq 0\,]$, condition (iv), given the following adjustments,

- *in (i): ψ_v is $(k/1)^{th}$-degree Ross more risk-averse than ψ_u, instead of the reverse;*
- *in (ii): (19a) is replaced by $\psi_v(c^R) = \lambda_\gamma \psi_u(c^R) + \gamma(c^R)$;*
- *in (iii): $\pi_{\psi_v}^R \geq \pi_{\psi_u}^R$, instead of the reverse.*

If $\phi_u'' \leq 0 \leq \phi_v''$ $[\, \phi_v'' \leq 0 \leq \phi_u''\,]$, both sets of conditions (i)–(iii) imply condition (iv).

---

esting because they immediately imply an ordering of θ_u^R and θ_v^R.

The equivalences among conditions (i), (ii), and (iii), and those among the three adjusted conditions, follow similarly to the ones in Theorem 1 and Liu, paying, in addition, only attention as to whether the agents increase or decrease saving. For illustration, we state in Appendix A the EU case, which, to our knowledge, has not been treated in the literature. The conclusion for $\theta_u^R \geq \theta_v^R \geq 0$ or $\theta_u^R \leq \theta_v^R \leq 0$ in condition (iv) is immediate from (18). All combinations of curvature constellations between ϕ_u and ϕ_v and the two sets of conditions (i)–(iii) in each formulation that Theorem 2 does not mention, again, lead either to an ambiguous effect or an effect in the wrong direction from the CE channel.

A particularity of the precautionary premium under return risk is that it can be positive or negative for the same risk preferences that are associated with positive precautionary saving under income risk. This fact highlights a weakness of this concept as a measure of the strength of the precautionary-saving motive: by referring to the change in total saving, θ^R nets out the underlying precautionary and substitution effects. These effects depend, respectively, on the intensities of the underlying precautionary and consumption-smoothing preferences. To better capture the strengths of the different preference dimensions, we turn next to constructing an alternative comparison using coefficient measures applying under income risk and return risk.

5 Representation by Preference Coefficients

Unlike premium measures, preference coefficients are widely used in empirical applications. We show now how to represent the comparisons of precautionary-saving motives using preference coefficients, starting from the concavity measure of Liu and Meyer (2013a).

To relate Ross more risk aversion and preference coefficients, Liu and Meyer refer to an expression similar to the Arrow-Pratt coefficient of absolute risk aversion. The difference is that marginal utility f' is always evaluated at a fixed value $x_0 \in [a, b]$. The monotonicity of f' and the negativity of f'' over $[a, b]$ admit the normalized measure

$$C_f(x; x_0) \equiv -\frac{f''(x)}{f'(x_0)}$$

based on which Ross more risk aversion can be equivalently represented when $x_0 = a$.[16]

To extend this measure to higher-order preferences, we rely on the measure of local $(n/m)^{th}$-degree absolute risk aversion that Liu and Meyer (2013b) define for an n-times differentiable function $f \in \mathcal{U}_{m-cv}^{[a,b]}$:

$$A_{(n/m)_f}(x) = \frac{(-1)^{n-1} f^{(n)}(x)}{(-1)^{m-1} f^{(m)}(x)}$$

Definition 3 (Generalized Liu and Meyer Measure) *For an n-times differentiable utility function $f(x) \in \mathcal{U}_{m-cv}^{[a,b]}$ with $n > m \geq 1$ and $[a,b] \subseteq D_x \subseteq \mathbb{R}_0^+$, the generalized Liu and Meyer measure of $(n/m)^{th}$-degree risk aversion is*

$$C_{(n/m)_f}(x;a) = (-1)^{n-m} \frac{f^{(n)}(x)}{f^{(m)}(a)} \tag{20}$$

This measure allows us to represent higher-order Ross more risk aversion ($m = 1$) and precautionary attitudes under EU ($m = 2$) alternatively as follows.

Lemma 2 *If ψ_u is more $((m+1)/m)^{th}$-degree risk-averse than ψ_v, i.e., $-\frac{\psi_u^{(m+1)}(x)}{\psi_u^{(m)}(x)} \geq -\frac{\psi_v^{(m+1)}(x)}{\psi_v^{(m)}(x)}$ for all $x \in [a,b] \subseteq D_x$, for $m = 1$ or 2, then condition (11) for $x_a, x_b \in [a,b] \subseteq D_x$ is equivalent to*

$$C_{(n/m)_{\psi_u}}(x;a) \geq C_{(n/m)_{\psi_v}}(x;a) \tag{21}$$

with $C_{(n/m)_f}(x;a)$ as defined in (20) for $f \in \{\psi_u, \psi_v\}$.

For $m = 1$, Lemma 2 complements the characterizations of n^{th}-degree Ross more risk aversion in Li (2009) and Denuit and Eeckhoudt (2010a) by a representation involving the generalized Liu and Meyer measures of $(n/1)^{th}$-degree risk aversion. This comparison requires agent u to be globally more Arrow-Pratt risk-averse than v also when $n > 2$.

For intertemporal choice under EU, Lemma 2 provides for $m = 2$ the conditions to compare the strengths of precautionary-saving motives in the face of risk increases

[16] Sufficiency results in this equivalence from the fact that the condition for Ross more risk aversion, $\frac{u''(x_a)}{v''(x_a)} \geq \lambda \geq \frac{u'(x_b)}{v'(x_b)}$ for all $x_a, x_b \in [a,b]$, implies for $x_b = a$ that $C_u(x;a) \geq C_v(x;a)$ for all $x_a \in [a,b]$. Necessity follows by integrating the latter inequality on both sides from a to x_b, yielding $\frac{u'(x_b)}{u'(a)} \leq \frac{v'(x_b)}{v'(a)}$ for all $x_b \in [a,b]$. The two latter inequalities, combined with $\lambda = \frac{u'(a)}{v'(a)}$, establish the link.

using preference coefficients. For example, it implies a representation of Theorem 3 in Liu for n-ℓ-MPSD using the $C_{(n/2)}$ coefficients for $(\psi_u, \psi_v) = (u_{eu}, v_{eu})$, $n = k+1$, and $k = \ell+1, \ldots, n$, under the additional condition that u_{eu} is globally more prudent than v_{eu}. An analogous representation holds, with analogous proof, for return risk (*cf.* Lemma 3 in Appendix A).

Lemma 2 allows us to state how the equivalent conditions in Theorems 1 and 2 can alternatively be represented based on preference coefficients.

Theorem 3 *Let u be more prudent than v, i.e.,* $-\frac{\psi_u'''(c)}{\psi_u''(c)} \geq -\frac{\psi_v'''(c)}{\psi_v''(c)}$ *for all* $c \in [a,b] \subseteq D_c$. *Then, if u is also more Arrow-Pratt risk-averse than v, i.e.,* $-\frac{\psi_u''(c)}{\psi_u'(c)} \geq -\frac{\psi_v''(c)}{\psi_v'(c)}$, *conditions (i)–(iii) in Theorem 1 for* $c = c_{y_2}$, *and in Theorem 2 for* $c = c_R$ *are equivalent to:*

$$C_{((k+1)/m)_{\psi_u}}(c;a) \geq C_{((k+1)/m)_{\psi_v}}(c;a) \quad \text{for } m = 1,2 \quad \text{and all} \quad k = \ell+1, \ldots, n \tag{22}$$

with $C_{((k+1)/2)_f}(c;a)$, $f \in \{\psi_u, \psi_v\}$, *as defined in* (20).

If, by contrast, v is more Arrow-Pratt risk-averse than u, then the equivalences with the adjustments in Theorems 1 and 2 are similarly equivalent to (22), *if in* (22) *it holds that* $C_{((k+1)/1)_{\psi_v}}(c;a) \geq C_{((k+1)/1)_{\psi_u}}(c;a)$, *instead of the reverse.*

Theorem 3, with Theorems 1 and 2, gives the conditions to predict from preference coefficients that one RU agent has a stronger precautionary-saving motive than another. In addition to the relation of second-order risk and intertemporal preferences, this requires to control for the order of the generalized Liu and Meyer measures $C_{((k+1)/1)}$ and $C_{((k+1)/2)}$, Arrow-Pratt risk aversion, and Kimball prudence. The two latter conditions are natural when comparing reactions to second-order risk under EU. Their relevance under higher-order risks reflects the fact that such risk changes necessarily entail positive variance.

The three theorems illustrate the increased complexity of comparing the preferences which underpin precautionary saving under RU as compared to EU. For EU agents, it is sufficient to inspect $((k+1)/2)^{th}$-degree risk aversion. For example, to exhibit a stronger precautionary-saving motive for an increase in downside risk of $\tilde{y}_2$ (a 3-2-MPSD shift) agent u_{eu} needs to be globally more prudent than agent v_{eu}, and their generalized Liu and Meyer temperance measures $C_{(4/2)}$ have to compare as $\frac{u''''_{eu}(c^{y_2})}{u''_{eu}(c^{y_2})} \geq \frac{v''''_{eu}(c^{y_2})}{v''_{eu}(c^{y_2})}$ for all $c^{y_2} \in [a,b]$. By contrast, comparing two RU agents requires to first inspect the relation of their second-order risk and consumption-smoothing preferences. If neither of the two has a preference for early risk resolution, so that ϕ_f is concave for $f \in \{u,v\}$, and $\phi''_u \leq \phi''_v$, risk preferences have to be compared for each risk-impact channel. An unambiguous conclusion for $\theta^{y_2}_u \geq \theta^{y_2}_v$ due to an increase in downside risk of $\tilde{y}_2$ follows if agent u is globally more Arrow-Pratt risk averse and globally more Kimball-prudent than agent v, and their $C_{(4/2)_\psi}$ measures compare as $\frac{\psi''''_u(c^{y_2})}{\psi''_u(c^{y_2})} \geq \frac{\psi''''_v(c^{y_2})}{\psi''_v(c^{y_2})}$ for all $c^{y_2} \in [a,b]$.

Lemma 2 and Theorem 3 underscore the flexible applicability of generalized Liu and Meyer measures for different kinds of preference comparisons, in static and dynamic contexts, for additive and multiplicative risks, and under EU and RU. Due to the coefficient-based approach, comparisons of the preferences in Theorems 1 and 2 become better amenable to empirical applications. In particular, the approach provides coherent metrics for either additive- or multiplicative-risk frameworks on the same conceptual grounds. The comparison of preference intensities between the two frameworks, however, hinges on the difficulty to compare risk sizes and equilibria in corresponding situations. Future quantitative applications can help to better judge the potential of this approach to clarify the preferences underpinning precautionary saving.

References

Attanasio, O. P. and Weber, G. Consumption and Saving: Models of Intertemporal Allocation and Their Implications for Public Policy. *Journal of Economic Literature*, 48(3): 693–751, 2010.

Bansal, R. and Yaron, A. Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles. *Journal of Finance*, 59(4): 1481–1509, 2004.

Barro, R. J. Rare Disasters, Asset Prices, and Welfare Costs. *American Economic Review*, 99(1): 243–264, 2009.

Binsbergen, J. V., Fernández-Villaverde, J., Koijen, R., and Rubio-Ramírez, J. The Term Structure of Interest Rates in a DSGE Model with Recursive Preferences. *Journal of Monetary Economics*, 59(7): 634–648, 2012.

Bommier, A., Chassagnon, A., and Le Grand, F. Comparative risk aversion: A formal approach with applications to savings behaviors. *Journal of Economic Theory*, 147(2): 1614–1641, 2012.

Bostian, A. A. and Heinzel, C. Precautionary Saving under Recursive Preferences. 2017. Working paper.

Briys, E., Eeckhoudt, L., and Loubergé, H. Endogenous Risks and the Risk Premium. *Theory and Decision*, 26(1): 37–46, 1989.

Capéraà, P. and Eeckhoudt, L. Delayed Risks and Risk Premiums. *Journal of Financial Economics*, 2(3): 309–320, 1975.

Chiu, W. H. Skewness Preference, Risk Aversion, and the Precedence Relations on Stochastic Changes. *Management Science*, 51(12): 1816–1828, 2005.

Choi, H., Lugauer, S., and Mark, N. C. Precautionary Saving of Chinese and U.S. Households. *Journal of Money, Credit and Banking*, 49(4): 635–661, 2017.

Denuit, M. M. and Eeckhoudt, L. Stronger measures of higher-order risk attitudes. *Journal of Economic Theory*, 145(5): 2027–2036, 2010a.

Denuit, M. M. and Eeckhoudt, L. A General Index of Absolute Risk Attitude. *Management Science*, 56(4): 712–715, 2010b.

Denuit, M. M. and Eeckhoudt, L. Risk attitudes and the value of risk transformations. *International Journal of Economic Theory*, 9(3): 245–254, 2013.

Drèze, J. H. and Modigliani, F. Consumption Decisions under Uncertainty. *Journal of Economic Theory*, 5(3): 308–335, 1972.

Dynan, K. E. How Prudent Are Consumers? *Journal of Political Economy*, 101(6): 1104–1113, 1993.

Eeckhoudt, L. Arrow-Pratt's Risk Aversion: 50 Years Later. 2014. Keynote lecture, Journées Internationales du Risque, Niort, France.

Eeckhoudt, L. and Schlesinger, H. Putting Risk in Its Proper Place. *American Economic Review*, 96(1): 280–289, 2006.

Eeckhoudt, L. and Schlesinger, H. Changes in risk and the demand for saving. *Journal of Monetary Economics*, 55(7): 1329–1336, 2008.

Eeckhoudt, L. and Schlesinger, H. On the utility premium of Friedman and Savage. *Economics Letters*, 105(1): 46–48, 2009.

Eisenhauer, J. G. Estimating Prudence. *Eastern Economic Journal*, 24(6): 379–392, 2000.

Ekern, S. Increasing Nth Degree Risk. *Economics Letters*, 6(4): 329–333, 1980.

Epstein, L. G. and Zin, S. E. Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework. *Econometrica*, 57(4): 937–969, 1989.

Epstein, L. G. and Zin, S. E. Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: An Empirical Analysis. *Journal of Political Economy*, 99(2): 263–286, 1991.

Epstein, L. G., Farhi, E., and Strzalecki, T. How Much Would You Pay to Resolve Long-Run Risk? *American Economic Review*, 104(9): 2680–2697, 2014.

Gollier, C. *The Economics of Risk and Time*. MIT Press, Cambridge, MA, 2001.

Gollier, C. *Pricing the Planet's Future: The Economics of Discounting in an Uncertain World*. 2013. Princeton University Press, Princeton, NJ.

Gomes, F. and Michaelides, A. Optimal Life-Cycle Asset Allocation: Understanding the Empirical Evidence. *Journal of Finance*, 60(2): 869–904, 2005.

Gourinchas, P.-O. and Parker, J. A. The Empirical Importance of Precautionary Saving. *American Economic Review*, 91(2): 406–412, 2001.

Hall, R. E. Intertemporal Substitution in Consumption. *Journal of Political Economy*, 96(2): 339–357, 1988.

Kihlstrom, R. E., Romer, D., and Williams, S. Risk Aversion with Random Initial Wealth. *Econometrica*, 49(4): 911–920, 1981.

Kimball, M. and Weil, P. Precautionary Saving and Consumption Smoothing across Time and Possibilities. *Journal of Money, Credit and Banking*, 41(2-3): 245–284, 2009.

Kimball, M. S. Precautionary Saving in the Small and in the Large. *Econometrica*, 58 (1): 53–73, 1990.

Kreps, D. M. and Porteus, E. L. Temporal Resolution of Uncertainty and Dynamic Choice Theory. *Econometrica*, 46(1): 185–200, 1978.

Langlais, E. A measure of the sensitivity of saving to interest rate uncertainty with non-expected preferences. *Economics Letters*, 48(3–4): 325–330, 1995.

Lee, J.-J. and Sawada, Y. The degree of precautionary saving: A reexamination. *Economics Letters*, 96(2): 196–201, 2007.

Li, J. Comparative higher-degree Ross risk aversion. *Insurance: Mathematics and Economics*, 45(3): 333–336, 2009.

Liu, L. Precautionary saving in the large: nth degree deteriorations in future income. *Journal of Mathematical Economics*, 52: 169–172, 2014.

Liu, L. and Meyer, J. Normalized measures of concavity and Ross's strongly more risk averse order. *Journal of Risk and Uncertainty*, 47(2): 185–198, 2013a.

Liu, L. and Meyer, J. Substituting one risk increase for another: A method for measuring risk aversion. *Journal of Economic Theory*, 148(6): 2706–2718, 2013b.

Lugilde, A., Bande, R., and Riveiro, D. Precautionary saving: a review of the theory and the evidence. 2017. MPRA Paper No. 77511.

Martin, I. W. Consumption-Based Asset Pricing with Higher Cumulants. *Review of Economic Studies*, 80(2): 745–773, 2013.

Menezes, C. E., Geiss, C., and Tressler, J. Increasing Downside Risk. *American Economic Review*, 70(5): 921–932, 1980.

Menezes, C. F. and Wang, X. H. Increasing outer risk. *Journal of Mathematical Economics*, 41(7): 875–886, 2005.

Merrigan, P. and Normandin, M. Precautionary Saving Motives: An Assessment from UK Time Series of Cross-Sections. *Economic Journal*, 106(438): 1193–1208, 1996.

Mirrlees, S. J., Adam, S., Besley, T., Blundell, R., Bond, S., Chote, R., Gammie, M., Johnson, P., Myles, G., and Poterba, J. *Dimensions of Tax Design. The Mirrlees Review.* Oxford University Press, Oxford, UK, 2010.

Nocetti, D. and Smith, W. T. Uncertainty, the Demand for Health Care, and Precautionary Saving. *B.E. Journal of Economic Analysis and Policy (Advances)*, 10(1): Article 75, 2010.

Nocetti, D. and Smith, W. T. Precautionary Saving and Endogenous Labor Supply with and without Intertemporal Expected Utility. *Journal of Money, Credit and Banking*, 43(7): 1475–1504, 2011a.

Nocetti, D. and Smith, W. T. Price uncertainty, saving, and welfare. *Journal of Economic Dynamics and Control*, 135(7): 1139–1149, 2011b.

Ross, S. A. Some Stronger Measures of Risk Aversion in the Small and the Large with Applications. *Econometrica*, 49(3): 621–638, 1981.

Rothschild, M. and Stiglitz, J. E. Increasing Risk I: A Definition. *Journal of Economic Theory*, 2(3): 225–243, 1970.

Selden, L. A New Representation of Preferences over "Certain x Uncertain" Consumption Pairs: The "Ordinal Certainty Equivalent" Hypothesis. *Econometrica*, 46(5): 1045–1060, 1978.

Selden, L. An OCE Analysis of the Effect of Uncertainty on Saving Under Risk Preference Independence. *Review of Economic Studies*, 46(1): 73–82, 1979.

Tallarini, T. D. Risk-sensitive real business cycles. *Journal of Monetary Economics*, 45 (3): 507–532, 2000.

Ventura, L. and Eisenhauer, J. G. Prudence and Precautionary Saving. *Journal of Economics and Finance*, 30(2): 155–168, 2006.

Wang, C., Wang, N., and Yang, J. Optimal consumption and savings with stochastic income and recursive utility. *Journal of Economic Theory*, 165: 292–331, 2016.

Weil, P. The Equity Premium Puzzle and the Risk-Free Rate Puzzle. *Journal of Monetary Economics*, 24(3): 401–421, 1989.

Weil, P. Nonexpected Utility in Macroeconomics. *Quarterly Journal of Economics*, 105 (1): 29–42, 1990.

Weil, P. Precautionary Savings and the Permanent Income Hypothesis. *Review of Economic Studies*, 60(2): 367–383, 1993.

Appendix

A Comparative Precautionary Saving under Return-Risk Increases and EU

This appendix treats the comparative precautionary-saving characterizations under return-risk increases in the EU special case.

Theorem 4 *Suppose $n \geq 2$ and $1 \leq \ell \leq n-1$, and consider two EU agents u_{eu}, v_{eu} with preferences in the class $\mathcal{U}_{k-cv}^{D_{c^R}}$ of all k-concave functions for $k = 1, 2$ and $\ell+1, \ldots, n+1$, who save the same amount under the reference return $\tilde{R}_l$ and increase [decrease] saving in response to return-risk increases in the n-ℓ-MPSD order. Then, (i)–(iii) are equivalent:*

(i) u_{eu} is $((k+1)/2)^{th}$-degree Ross more risk-averse than v_{eu} for $k = \ell+1, \ldots, n$.

(ii) There exist $\lambda > 0$ and $\eta'(c^R)$ with $\eta''(c^R) \geq 0$ and $\eta'(c^R) \in \mathcal{U}_{(k+1)-cv}^{D_{c^R}}$ for $k = \ell+1, \ldots, n$ and all $c^R \in D_{c^R} \subset \mathbb{R}_0^+$ such that

$$u'_{eu}(c^R) = \lambda v'_{eu}(c^R) + \eta'(c^R) \tag{23}$$

(iii) $\theta_{u_{eu}}^R \geq \theta_{v_{eu}}^R \geq 0$ $\left[\, \theta_{u_{eu}}^R \leq \theta_{v_{eu}}^R \leq 0 \,\right]$ for all $\tilde{R}_l, \tilde{R}_h$ with $\tilde{R}_l \succeq_{n-\ell-MPSD} \tilde{R}_h$ and θ_ψ^R as defined in equation (17) for $\psi \in \{u_{eu}, v_{eu}\}$.

Proof. (i) $\Rightarrow$ (ii) is analogous to Liu.

For the other two steps of the proof, consider first the case where both agents increase saving in response to a given return-risk increase, so that $\theta_{u_{eu}}^R, \theta_{v_{eu}}^R \geq 0$.

(ii) $\Rightarrow$ (iii). Thanks to (17) for $\psi \in \{u_{eu}, v_{eu}\}$ and the concavity of u_{eu},

$$\theta_{u_{eu}}^R \geq \theta_{v_{eu}}^R \;\Leftrightarrow\; E[u'_{eu}(y_2 + s(\tilde{R}_l - \theta_{u_{eu}}^R))\tilde{R}_l] \geq E[u'_{eu}(y_2 + s(\tilde{R}_l - \theta_{v_{eu}}^R))\tilde{R}_l] \tag{24}$$

With (23), $\eta'(c^R) \in \mathcal{U}_{(k+1)-cv}^{D_{c^R}}$ for $k = \ell+1, \ldots, n$, and $\eta''(c^R) \geq 0$, (24) holds because

$$E[u'_{eu}(y_2 + s(\tilde{R}_l - \theta_{u_{eu}}^R))\tilde{R}_l] \;=\; E[u'_{eu}(y_2 + s\tilde{R}_h)\tilde{R}_h]$$

$$
\begin{aligned}
&= \lambda E[v'_{eu}(y_2 + s\tilde{R}_h)\tilde{R}_h] + E[\eta'(y_2 + s\tilde{R}_h)\tilde{R}_h] \\
&\geq \lambda E[v'_{eu}(y_2 + s\tilde{R}_h)\tilde{R}_h] + E[\eta'(y_2 + s\tilde{R}_l)\tilde{R}_l] \\
&= \lambda E[v'_{eu}(y_2 + s(\tilde{R}_l - \theta^R_{v_{eu}}))\tilde{R}_l] + E[\eta'(y_2 + s\tilde{R}_l)\tilde{R}_l] \\
&\geq \lambda E[v'_{eu}(y_2 + s(\tilde{R}_l - \theta^R_{v_{eu}}))\tilde{R}_l] + E[\eta'(y_2 + s(\tilde{R}_l - \theta^R_{v_{eu}}))\tilde{R}_l] \\
&= E[u'_{eu}(y_2 + s(\tilde{R}_l - \theta^R_{v_{eu}}))\tilde{R}_l],
\end{aligned}
$$

where the first line is equation (17) for $\psi = u_{eu}$, the second line applies (23), the first inequality follows from Lemma 1 for $f' = \eta'$ as applied to a return-risk increases in the n-ℓ-MPSD order, i.e., when conditions (7) for $u_{eu} = \eta$ are fulfilled for $k = \ell + 1, \ldots, n$ (as it is the case when saving increases), the fourth line applies (17) for $\psi = v_{eu}$, the second inequality follows from $\eta''(c^R) \geq 0$ and $\theta^R_{v_{eu}} \geq 0$, and the last line applies again (23).

(iii) ⇒ (i). $\theta^R_{u_{eu}} \geq \theta^R_{v_{eu}} \geq 0$ for all $\tilde{R}_l, \tilde{R}_h$ such that $\tilde{R}_l \succeq_{n-\ell-MPSD} \tilde{R}_h$ and θ^R_ψ defined as in (17) for $\psi \in \{u_{eu}, v_{eu}\}$ implies that $\theta^R_{u_{eu}} \geq \theta^R_{v_{eu}} \geq 0$ for all $\tilde{R}_l, \tilde{R}_h$ such that $\tilde{R}_h$ is a k^{th}-degree Ekern risk increase over $\tilde{R}_l$ for all $k = \ell + 1, \ldots, n$. Based on Definition 2 and an argument analogous to Liu's, this implies that u_{eu} is $((k+1)/2)^{th}$-degree Ross more risk averse than v_{eu} for all $k = \ell + 1, \ldots, n$.

When both agents decrease saving in response to the return-risk increase, so that $\theta^R_{u_{eu}}, \theta^R_{v_{eu}} \leq 0$, (ii) ⇒ (iii) follows as before, only the direction of the inequalities is reversed. For (iii) ⇒ (i), the proof starting from $\theta^R_{u_{eu}} \leq \theta^R_{v_{eu}} \leq 0$ is analogous. ■

Using the generalized Liu and Meyer measures of Definition 3, the comparative precautionary saving conditions in Theorem 4 have an equivalent representation as follows.

Lemma 3 *If u_{eu} is more prudent than v_{eu}, i.e., $-\frac{u'''_{eu}(c^R)}{u''_{eu}(c^R)} \geq -\frac{v'''_{eu}(c^R)}{v''_{eu}(c^R)}$ for all $c^R \in [a,b] \subset D_{c^R}$, then conditions (i)–(iii) in Theorem 4 are equivalent to:*

$$
C_{((k+1)/2)_{u_{eu}}}(c^R; a) \geq C_{((k+1)/2)_{v_{eu}}}(c^R; a) \quad \text{for all} \quad k = \ell + 1, \ldots, n
$$

with $C_{((k+1)/2)_f}(c^R; a)$ as defined in (20) for $f \in \{u_{eu}, v_{eu}\}$.

B Proof of Lemma 2

Sufficiency. Given condition (11) for $x_a, x_b \in [a,b] \subseteq D_x$, there exists for $m \in \{1,2\}$ and all $x_a, x_b \in [a,b] \subseteq D_x$ a $\lambda > 0$ such that

$$\frac{\psi_u^{(n)}(x_a)}{\psi_v^{(n)}(x_a)} \geq \lambda \geq \frac{\psi_u^{(m)}(x_b)}{\psi_v^{(m)}(x_b)}\,. \tag{25}$$

Let $x_b = a$. Then, for all $x \in [a,b]$,

$$\frac{\psi_u^{(n)}(x)}{\psi_v^{(n)}(x)} \geq \frac{\psi_u^{(m)}(a)}{\psi_v^{(m)}(a)} \quad \Leftrightarrow \quad (-1)^{n-m}\frac{\psi_u^{(n)}(x)}{\psi_u^{(m)}(a)} \geq (-1)^{n-m}\frac{\psi_v^{(n)}(x)}{\psi_v^{(m)}(a)}\,.$$

Necessity. Assume (20), so that, for $x = x_a$,

$$(-1)^{n-m}\frac{\psi_u^{(n)}(x_a)}{\psi_u^{(m)}(a)} \geq (-1)^{n-m}\frac{\psi_v^{(n)}(x_a)}{\psi_v^{(m)}(a)} \quad \Leftrightarrow \quad \frac{\psi_u^{(n)}(x_a)}{\psi_v^{(n)}(x_a)} \geq \frac{\psi_u^{(m)}(a)}{\psi_v^{(m)}(a)}\,.$$

The first inequality in (25) arises by setting $\lambda \equiv \frac{\psi_u^{(m)}(a)}{\psi_v^{(m)}(a)}$. The second holds because $\frac{\psi_u^{(m)}(a)}{\psi_v^{(m)}(a)}$ decreases with its argument if, for all $x \in [a,b] \subseteq D_x$, $-\frac{\psi_u^{(m+1)}(x)}{\psi_u^{(m)}(x)} \geq -\frac{\psi_v^{(m+1)}(x)}{\psi_v^{(m)}(x)}$. ■